Optimal Quantum State Discrimination with Fixed Measurements

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***Abstract*—Given an unknown quantum state described by one of two possible density operators, the Helstrom bound provides the minimum discrimination error probability (DEP) by optimizing over all possible quantum measurements. However, it is unrealistic to implement arbitrary measurements in practice due to physical limitations of measurement apparatuses. This paper considers a quantum state discrimination scenario where a fixed measurement apparatus is available. In this setting, we advocate the use of quantum pre-processing (QPP) to realize effectively different measurements from that of the fixed apparatus. Applying optimal QPP prior to measurement with the fixed apparatus allows one to minimize the DEP. This paper derives the minimum DEP, determines the QPP required to achieve it, and provides necessary and sufficient conditions for this minimum DEP with optimal QPP to coincide with the Helstrom bound.**

***Index Terms*—Quantum state discrimination, quantum pre-processing, Helstrom bound, quantum communications, quantum systems.**

I. Introduction

A foundational result in quantum information is the Helstrom bound [1], [2, pp. 106-108], which states that the minimum discrimination error probability (DEP) in binary quantum state discrimination is characterized entirely by the trace distance between the two density operators to be distinguished [3]–[6]. This bound is achieved by minimizing the DEP over all possible quantum measurements. However, if measurement capabilities are constrained [7]–[9], then the Helstrom bound is unachievable in general. Therefore, researchers are often interested in measurement systems that do not necessarily achieve the Helstrom bound [10]–[13]. This paper considers binary quantum state discrimination when a fixed measurement apparatus is available. To realize effectively different measurement systems, quantum pre-processing (QPP) is applied to the unknown state prior to measurement with the fixed apparatus. The goal is to derive the minimum achievable DEP in this fixed measurement setting, i.e., the tightest possible DEP bound. By construction, the minimum DEP depends directly on the available experimental setup and QPP capabilities.

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To the best of our knowledge, the device-specific bounds on the achievable DEP developed in this paper chart new territory with respect to the existing literature. The closest related existing works considered quantum state discrimination with measurements belonging to *restricted families*, such as local measurements, separable measurements, positive under partial transposition measurements, and measurements implementable by local operations and classical communication (LOCC) [14]–[17]. These types of restricted families arise, e.g., in quantum networks. In part, these restricted families are of interest because there exist orthogonal states which they cannot reliably distinguish [14]. The paper [15] established a mathematical framework for studying restricted families of measurements by introducing a semi-norm dependent on the restricted family in consideration. This semi-norm replaces the trace distance in the Helstrom bound to give the distinguishability of quantum states. Other works studied the use of restricted measurement families in adversarial quantum hypothesis testing [17] as well as in channel discrimination [18]. For a complete literature review of quantum state discrimination see [19]; for an assortment on quantum communications and networking see [20]–[30].

While restricted families of measurements are interesting in their own right, the present paper is concerned with systems where a fixed measurement apparatus is available and QPP is applied prior to measurement. This setting is crucial to understanding the fundamental limits of, among others, quantum computing with imperfect (non-projective) measurements. The contributions of this paper are as follows: we

- derive a lower bound on the DEP achievable when QPP is used prior to measurement;
- prove that this bound is achievable by optimizing over QPP and one's decision rule;
- provide closed-form expressions for optimal QPP achieving the minimum DEP; and
- give conditions for the minimum DEP with optimal QPP to coincide with the Helstrom bound.

To put these results into action, we quantify the optimal DEP of a quantum computer-enabled optical receiver for coherent state binary phase-shift keying (BPSK) communications [31].

Notation: Random variables are displayed in sans serif, upright fonts; their realizations in serif, italic fonts. Matri-

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ces are denoted by bold uppercase letters. For example, a random variable and its realization are denoted by x and x; a random matrix and its realization are denoted by $\mathbf{X}$ and $\boldsymbol{X}$, respectively. The Hermitian conjugate, rank, and trace of a matrix $\boldsymbol{X}$ are denoted $\boldsymbol{X}^\dagger$, $\mathrm{rank}\{\boldsymbol{X}\}$, and $\mathrm{tr}\{\boldsymbol{X}\}$, respectively. The m-by-m identity matrix is denoted by $\boldsymbol{I}_m$; the subscript is removed when the dimension of the matrix is clear from the context. The m-by-n matrix of zeros is denoted by $\mathbf{0}_{m\times n}$. For a square matrix $\boldsymbol{X}$, the notation $\boldsymbol{X} \succcurlyeq \mathbf{0}$ means that $\boldsymbol{X}$ is positive semidefinite, $\lambda_k^{\uparrow}(\boldsymbol{X})$ denotes the kth smallest eigenvalue of $\boldsymbol{X}$, and $\lambda_k^{\downarrow}(\boldsymbol{X})$ denotes the kth largest eigenvalue of $\boldsymbol{X}$. The unitary group of order d is defined as $\mathcal{U}(d) = \{\boldsymbol{U} \in \mathbb{C}^{d\times d} : \boldsymbol{U}^\dagger\boldsymbol{U} = \boldsymbol{U}\boldsymbol{U}^\dagger = \boldsymbol{I}\}$. The Kronecker product is denoted $\otimes$ and the mth Kronecker power of $\boldsymbol{X}$ is denoted $\boldsymbol{X}^{\otimes m}$. The Kraus rank of a linear map K on the set of density matrices is denoted $\mathrm{krank}\{K\}$. The set of natural, real, and complex numbers are denoted $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Let $m, n \in \mathbb{N}\cup\{0\}$ with $m \leqslant n$; $\mathbb{N}_m^n$ denotes the set $\{m, m+1, ..., n\}$. All other sets are denoted by calligraphic font as in $\mathcal{X}$. The indicator function is defined as $\mathbb{1}_{\mathcal{X}}(x) = 1$ if $x \in \mathcal{X}$ and $\mathbb{1}_{\mathcal{X}}(x) = 0$ otherwise.

II. System Model

The state of a finite d-dimensional quantum system can be represented by a density matrix in $\mathbb{C}^{d\times d}$. We will use the terminology "quantum state" and "the density matrix describing the quantum state" interchangeably. This section first revisits quantum state discrimination, then presents a model for QPP-based quantum state discrimination in the fixed measurement scenario.

A. Preliminaries on quantum state discrimination

In Bayesian binary quantum state discrimination, the state $\mathbf{\Xi}$ is random and takes on one of two possible distinct values, denoted by the density operators $\boldsymbol{\Xi}_0$ and $\boldsymbol{\Xi}_1$. Let $p_0 \triangleq \mathbb{P}\{\mathbf{\Xi} = \boldsymbol{\Xi}_0\}$ and $p_1 \triangleq \mathbb{P}\{\mathbf{\Xi} = \boldsymbol{\Xi}_1\}$ denote the "prior" probabilities of $\boldsymbol{\Xi}_0$ and $\boldsymbol{\Xi}_1$, respectively. In the formalism of hypothesis testing, let

$$\begin{aligned} H_0: &\quad \mathbf{\Xi} = \boldsymbol{\Xi}_0 \\ H_1: &\quad \mathbf{\Xi} = \boldsymbol{\Xi}_1 . \end{aligned} \tag{1}$$

The goal is to decide which of the two states $\boldsymbol{\Xi}_0$ and $\boldsymbol{\Xi}_1$ the system is in. Note that this mathematical formulation contains the multi-copy discrimination problem as a special case. For instance, suppose that one wishes to distinguish two quantum states, $\boldsymbol{\Upsilon}_0$ or $\boldsymbol{\Upsilon}_1$, based on $m \in \mathbb{N}$ independent copies of the unknown state. Taking $\boldsymbol{\Xi}_0 = \boldsymbol{\Upsilon}_0^{\otimes m}$ and $\boldsymbol{\Xi}_1 = \boldsymbol{\Upsilon}_1^{\otimes m}$ recasts the multi-copy problem into the mathematical form (1) considered herein. The traditional model of quantum state discrimination is as follows.

- To infer the true value of $\mathbf{\Xi}$, one measures the state. The measurement is characterized by a positive operator-valued measure (POVM) system $\mathcal{M} = \{\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_N\}$ where $N \geqslant 2$, $\boldsymbol{M}_y \succcurlyeq \mathbf{0}$ for $y \in \mathbb{N}_1^N$, and $\sum_{y=1}^N \boldsymbol{M}_y = \boldsymbol{I}$. The outcome of the measurement is a random variable y taking values in $\mathbb{N}_1^N$ and distributed according to Born's rule

$$\mathbb{P}\{\,\mathsf{y} = y \,|\, \mathbf{\Xi} = \boldsymbol{\Xi}_0\,\} = \mathrm{tr}\{\boldsymbol{M}_y\boldsymbol{\Xi}_0\} \tag{2a}$$
$$\mathbb{P}\{\,\mathsf{y} = y \,|\, \mathbf{\Xi} = \boldsymbol{\Xi}_1\,\} = \mathrm{tr}\{\boldsymbol{M}_y\boldsymbol{\Xi}_1\} . \tag{2b}$$

- Once the state $\mathbf{\Xi}$ is measured, the measurement outcome y is mapped to a decision $\hat{H}$ about the true hypothesis using a *decision rule* $\mathbb{1}_{\mathcal{S}}(\cdot) : \mathbb{N}_1^N \to \{0, 1\}$.[1] A decision rule is parameterized by a *decision region* $\mathcal{S} \subseteq \mathbb{N}_1^N$. If $\mathbb{1}_{\mathcal{S}}(y) = 0$ then $\hat{H} = H_0$, and if $\mathbb{1}_{\mathcal{S}}(y) = 1$ then $\hat{H} = H_1$.

The performance of a particular quantum state discrimination scheme (comprised of the POVM $\mathcal{M}$ and the decision region $\mathcal{S}$) is characterized by the probability of inferring the wrong hypothesis, i.e., the DEP given by

$$P_{\mathrm{e}}^{\mathcal{M}}(\mathcal{S}) = p_0 \sum_{y\in\mathcal{S}} \mathrm{tr}\{\boldsymbol{M}_y\boldsymbol{\Xi}_0\} + p_1 \sum_{y\in\mathcal{S}^{\mathrm{c}}} \mathrm{tr}\{\boldsymbol{M}_y\boldsymbol{\Xi}_1\} \tag{3}$$

where $\mathcal{S}^{\mathrm{c}} \triangleq \mathbb{N}_1^N \setminus \mathcal{S}$. The expression (3) can be written in a more insightful form by noting that the decision rule $\mathbb{1}_{\mathcal{S}}(\cdot)$ induces from $\mathcal{M}$ the binary POVM

$$\mathcal{M}(\mathcal{S}) \triangleq \{\, \boldsymbol{M}(\mathcal{S}) \,,\, \boldsymbol{I} - \boldsymbol{M}(\mathcal{S}) \,\} \tag{4}$$

where $\boldsymbol{M}(\mathcal{S}) \triangleq \sum_{y\in\mathcal{S}} \boldsymbol{M}_y$ and $\boldsymbol{I} - \boldsymbol{M}(\mathcal{S}) = \sum_{y\in\mathcal{S}^{\mathrm{c}}} \boldsymbol{M}_y$. The DEP reduces to

$$P_{\mathrm{e}}^{\mathcal{M}}(\mathcal{S}) = p_0\,\mathrm{tr}\{\boldsymbol{M}(\mathcal{S})\boldsymbol{\Xi}_0\} + p_1\,\mathrm{tr}\{[\boldsymbol{I} - \boldsymbol{M}(\mathcal{S})]\boldsymbol{\Xi}_1\} \tag{5a}$$
$$= p_1 - \mathrm{tr}\{\boldsymbol{M}(\mathcal{S})\underbrace{[\,p_1\boldsymbol{\Xi}_1 - p_0\boldsymbol{\Xi}_0\,]}_{\triangleq \boldsymbol{Z}}\} \tag{5b}$$

where $\boldsymbol{Z} \in \mathbb{C}^{d\times d}$. Hence, to minimize the DEP, the quantity $\mathrm{tr}\{\boldsymbol{M}(\mathcal{S})\boldsymbol{Z}\}$ should be as large as possible. An optimal decision region $\mathcal{S}^\star \subseteq \mathbb{N}_1^N$ is defined according to the following rule that maximizes $\mathrm{tr}\{\boldsymbol{M}(\mathcal{S})\boldsymbol{Z}\}$: for every $y \in \mathbb{N}_1^N$, $y \in \mathcal{S}^\star$ if and only if $\mathrm{tr}\{\boldsymbol{M}_y\boldsymbol{Z}\} \geqslant 0$.[2]

Remark 1: The problem is trivial in the cases where $\boldsymbol{Z}$ is negative semidefinite or positive semidefinite: an optimal decision rule is to always decide $\boldsymbol{\Xi}_0$ or $\boldsymbol{\Xi}_1$ regardless of the measurement outcome. We exclude these scenarios and consider the interesting case where $\boldsymbol{Z}$ is indefinite. □

A seminal result of quantum information states that the DEP in (5) satisfies the condition

$$P_{\mathrm{e}}^{\mathcal{M}}(\mathcal{S}) \geqslant \frac{1}{2}\big(1 - \|\boldsymbol{Z}\|_1\big) \triangleq \breve{P}_{\mathrm{e}} \tag{6}$$

which is known as the Helstrom bound [2, pp. 106-108]. It was shown that equality in (6) can be achieved if one can implement a measurement characterized by the binary POVM $\mathcal{M}^\star = \{\boldsymbol{M}^\star, \boldsymbol{I} - \boldsymbol{M}^\star\}$ where where $\boldsymbol{M}^\star$ is the projector onto the range of $[\boldsymbol{Z}]_+ \triangleq (\boldsymbol{Z} + \sqrt{\boldsymbol{Z}^2})/2$.[3] The optimal decision rule associated with the POVM $\mathcal{M}^\star$ is to decide $\hat{H} = H_1$ if

[1] For simplicity, only deterministic decision rules are considered in this paper. It can be shown that random decision rules perform no better than deterministic ones in the context of this paper [2], [32].

[2] Notice that if $\mathrm{tr}\{\boldsymbol{M}_y\,\boldsymbol{Z}\} = 0$ for some $y \in \mathbb{N}_1^N$ then $\mathcal{S}^\star$ is not a unique optimal decision region.

[3] Since $\boldsymbol{Z}$ is Hermitian, $\boldsymbol{Z}^2$ has a unique positive square root [33, p. 253].

the outcome associated with $\boldsymbol{M}^\star$ is observed and to decide $\hat{H} = H_0$ otherwise. While the Helstrom bound provides the ultimate quantum limit on state discrimination, as we will see shortly, there are systems of interest where the lower bound (6) is loose.

B. Quantum state discrimination with fixed measurements

Suppose that we only have access to a physical measurement apparatus that implements a fixed measurement, characterized by the POVM $\mathcal{M}$. As the POVM $\mathcal{M}$ is unlikely conducive to minimizing the DEP, we introduce the concept of QPP which is used to alter the distribution of the measurement outcome y, encoding more information about the underlying hypothesis. With QPP, the traditional model of quantum state discrimination is reformulated as follows.

- Prior to measuring the unknown state $\boldsymbol{\Xi}$ with the available measurement system, pre-process the state $\boldsymbol{\Xi}$. This action, referred to as QPP, is mathematically modeled as passing the unknown state $\boldsymbol{\Xi}$ through a quantum channel described by completely positive and trace-preserving (CPTP) mapping $K(\cdot) : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$.

After QPP, the state $K(\boldsymbol{\Xi})$ is measured using the measurement characterized by $\mathcal{M}$ and the decision is made according to a rule $\mathbb{1}_{\mathcal{S}}(\mathsf{y})$ as before. The effect of QPP is to alter the distribution of the measurement outcome y taking values in $\mathbb{N}_1^N$, namely (2) becomes

$$\mathbb{P}\{\, \mathsf{y} = y \,|\, \boldsymbol{\Xi} = \boldsymbol{\varXi}_0 \,\} = \mathrm{tr}\{\boldsymbol{M}_y K(\boldsymbol{\varXi}_0)\} \tag{7a}$$
$$\mathbb{P}\{\, \mathsf{y} = y \,|\, \boldsymbol{\Xi} = \boldsymbol{\varXi}_1 \,\} = \mathrm{tr}\{\boldsymbol{M}_y K(\boldsymbol{\varXi}_1)\}\,. \tag{7b}$$

The DEP associated with QPP $K(\cdot)$, the POVM $\mathcal{M}$, and the decision rule $\mathbb{1}_{\mathcal{S}}(\cdot)$ is

$$P_{\mathrm{e}}^{\mathcal{M}}(\mathcal{S}; K(\cdot)) = p_1 - \mathrm{tr}\{\boldsymbol{M}(\mathcal{S}) K(\boldsymbol{Z})\}\,. \tag{8}$$

This expression follows from (5) using the linearity of quantum channels.

For minimizing DEP (8), the quantity $\mathrm{tr}\{\boldsymbol{M}(\mathcal{S})K(\boldsymbol{Z})\}$ should be as large as possible. Whereas only $\mathcal{S}$ is to be optimized in the traditional model of quantum state discrimination, now both the QPP $K(\cdot)$ and the decision region $\mathcal{S}$ are to be optimized. The goal is to solve the optimization problem

$$\mathscr{P}_1: \quad \underset{\mathcal{S}\subseteq\mathbb{N}_1^N}{\text{maximize}} \quad \underset{K(\cdot)\in\mathcal{K}}{\text{maximize}} \quad \mathrm{tr}\{\boldsymbol{M}(\mathcal{S})K(\boldsymbol{Z})\}$$

where $\mathcal{K}$ is the set of feasible QPP (i.e., the physical transformations that one may perform on the unknown state $\boldsymbol{\Xi}$).[4] In what follows, the class of Kraus rank-constrained QPP is considered. Thus, the feasible QPP sets are of the form

$$\mathcal{K}(N_{\mathrm{k}}) \triangleq \{K(\cdot) \,:\, \mathrm{krank}\{K(\cdot)\} \leqslant N_{\mathrm{k}}\} \tag{9}$$

with $N_{\mathrm{k}} \in \mathbb{N}_1^{d^2}$. This type of feasible QPP arises in many cases of interest. For instance, the effects of a quantum circuit with a limited number of ancilla on a target state may be represented in this way, as will be demonstrated in Section V. In $\mathscr{P}_1$, the inner optimization problem concerns optimal QPP and the outer optimization problem concerns optimal decision making. Section III solves the optimal QPP problem in closed-form.

[4]It can be shown that the order of maximization in $\mathscr{P}_1$ is irrelevant. In particular, one can first maximize over QPP and then over the decision rule, vice versa, or simultaneously. Each configuration gives the same maximum.

III. Optimal Quantum Pre-Processing (QPP)

The following theorem provides the exact performance of the optimal rank-constrained QPP. How to achieve this optimal performance is discussed after the theorem's proof. For notational simplicity, $\boldsymbol{M}$ is written in place of $\boldsymbol{M}(\mathcal{S})$. Also, let $\nu_+ \in \mathbb{N}_0^d$ and $\nu_- \in \mathbb{N}_0^d$ denote the number of strictly positive and strictly negative eigenvalues of $\boldsymbol{Z}$, respectively.

Theorem 1: The optimal value of the rank-constrained QPP optimization problem

$$\mathscr{P}_2: \quad \underset{K(\cdot)\in\mathcal{K}(N_{\mathrm{k}})}{\text{maximize}} \quad \mathrm{tr}\{\boldsymbol{M} K(\boldsymbol{Z})\}$$

is given by (10b) at the top of the following page. If $N_{\mathrm{k}} \geqslant \max\{\nu_+, \nu_-\}$, then the optimal value of $\mathscr{P}_2$ reduces to

$$\frac{\lambda_1^{\downarrow}(\boldsymbol{M}) - \lambda_1^{\uparrow}(\boldsymbol{M})}{2}\left(\|\boldsymbol{Z}\|_1 + (p_1 - p_0)\right). \tag{11}$$

□

Proof: First consider (10b). Any QPP of Kraus rank at most N_{k} can be written as $K(\cdot) : \boldsymbol{Z} \mapsto \sum_{n=1}^{N_{\mathrm{k}}} \boldsymbol{V}_n \boldsymbol{Z} \boldsymbol{V}_n^\dagger$ with $\boldsymbol{V}_n \in \mathbb{C}^{d\times d}$ satisfying $\sum_{n=1}^{N_{\mathrm{k}}} \boldsymbol{V}_n^\dagger \boldsymbol{V}_n = \boldsymbol{I}_d$. Because of this relation, one may select a set of matrices $\boldsymbol{W}_n \in \mathbb{C}^{d\times(N_{\mathrm{k}}-1)d}$ such that the block matrix

$$\boldsymbol{X} = \left[\begin{array}{cccc} \boldsymbol{V}_1^\dagger & \boldsymbol{V}_2^\dagger & \cdots & \boldsymbol{V}_{N_{\mathrm{k}}}^\dagger \\ \hline \boldsymbol{W}_1^\dagger & \boldsymbol{W}_2^\dagger & \cdots & \boldsymbol{W}_{N_{\mathrm{k}}}^\dagger \end{array}\right] \in \mathbb{C}^{N_{\mathrm{k}}d\times N_{\mathrm{k}}d} \tag{12}$$

is unitary. Similarly, one may partition any unitary $\boldsymbol{X} \in \mathcal{U}(N_{\mathrm{k}}d)$ as in (12) to obtain a QPP with Kraus rank at most N_{k}.

Defining $\boldsymbol{A} \triangleq \boldsymbol{I}_{N_{\mathrm{k}}} \otimes \boldsymbol{M}$ and

$$\boldsymbol{B} \triangleq \left[\begin{array}{c|c} \boldsymbol{Z} & \boldsymbol{0}_{d\times(N_{\mathrm{k}}-1)d} \\ \hline \boldsymbol{0}_{(N_{\mathrm{k}}-1)d\times d} & \boldsymbol{0}_{(N_{\mathrm{k}}-1)d\times(N_{\mathrm{k}}-1)d} \end{array}\right] \tag{13}$$

it is straightforward to show, regardless of the particular values of the $\boldsymbol{W}_n$'s, that

$$\mathrm{tr}\{\boldsymbol{M}K(\boldsymbol{Z})\} = \mathrm{tr}\{\boldsymbol{X}\boldsymbol{A}\,\boldsymbol{X}^\dagger\boldsymbol{B}\}\,. \tag{14}$$

The maximum of the right hand side of (14) is known [34, p. 772], namely

$$\underset{\boldsymbol{X}\in\mathcal{U}(N_{\mathrm{k}}d)}{\text{maximum}}\ \mathrm{tr}\{\boldsymbol{X}\boldsymbol{A}\,\boldsymbol{X}^\dagger\boldsymbol{B}\} = \sum_{j=1}^{N_{\mathrm{k}}d} \lambda_j^{\downarrow}(\boldsymbol{A})\,\lambda_j^{\downarrow}(\boldsymbol{B})\,. \tag{15}$$

Given that $\boldsymbol{B}$ has at least $(N_{\mathrm{k}}-1)d$ zero eigenvalues, (10a) holds. The proof of (10b) is completed by noting that $\boldsymbol{Z}$ has $(d - \nu_+ - \nu_-)$ zero eigenvalues so that $\lambda_j^{\downarrow}(\boldsymbol{Z}) = 0$ if $\nu_+ < j \leqslant d - \nu_-$.

To prove (11) observe that the eigenvalues of $\boldsymbol{I}_{N_{\mathrm{k}}} \otimes \boldsymbol{M}$ are those of $\boldsymbol{M}$ repeated N_{k} times. If $N_{\mathrm{k}} \geqslant \max\{\nu_+, \nu_-\}$ then the ν_+ largest eigenvalues of $\boldsymbol{I}_{N_{\mathrm{k}}} \otimes \boldsymbol{M}$ are all exactly $\lambda_1^{\downarrow}(\boldsymbol{M})$; the

$$\underset{K(\cdot)\in\mathcal{K}(N_\mathrm{k})}{\text{maximum}} \ \operatorname{tr}\{\boldsymbol{M}K(\boldsymbol{Z})\} = \sum_{j=1}^{\nu_+} \lambda_j^{\downarrow}(\boldsymbol{Z})\lambda_j^{\downarrow}(\boldsymbol{I}_{N_\mathrm{k}}\otimes\boldsymbol{M}) + \sum_{j=\nu_+ +(N_\mathrm{k}-1)d+1}^{N_\mathrm{k}d} \lambda_{j-(N_\mathrm{k}-1)d}^{\downarrow}(\boldsymbol{Z})\,\lambda_j^{\downarrow}(\boldsymbol{I}_{N_\mathrm{k}}\otimes\boldsymbol{M}) \tag{10a}$$

$$= \sum_{j=1}^{\nu_+} \lambda_j^{\downarrow}(\boldsymbol{Z})\lambda_j^{\downarrow}(\boldsymbol{I}_{N_\mathrm{k}}\otimes\boldsymbol{M}) + \sum_{j=N_\mathrm{k}d-\nu_-+1}^{N_\mathrm{k}d} \lambda_{j-(N_\mathrm{k}-1)d}^{\downarrow}(\boldsymbol{Z})\,\lambda_j^{\downarrow}(\boldsymbol{I}_{N_\mathrm{k}}\otimes\boldsymbol{M}) \tag{10b}$$

ν_- smallest eigenvalues of $\boldsymbol{I}_{N_\mathrm{k}}\otimes\boldsymbol{M}$ are all exactly $\lambda_1^{\uparrow}(\boldsymbol{M})$. Hence, when $N_\mathrm{k}\geqslant\max\{\nu_+,\nu_-\}$, (10b) can be simplified to

$$\underset{K(\cdot)\in\mathcal{K}(\max\{\nu_+,\nu_-\})}{\text{maximum}} \ \operatorname{tr}\{\boldsymbol{M}K(\boldsymbol{Z})\}$$
$$= \lambda_1^{\downarrow}(\boldsymbol{M})\sum_{j=1}^{\nu_+}\lambda_j^{\downarrow}(\boldsymbol{Z}) + \lambda_1^{\uparrow}(\boldsymbol{M})\sum_{j=d-\nu_-+1}^{d}\lambda_j^{\downarrow}(\boldsymbol{Z})\,. \tag{16}$$

On the other hand, note that $\boldsymbol{Z}=p_1\boldsymbol{\Xi}_1-p_0\boldsymbol{\Xi}_0$ and $\operatorname{tr}\{\boldsymbol{Z}\}=p_1-p_0$. Therefore, the eigenvalues of $\boldsymbol{Z}$ sum to (p_1-p_0) and the equality

$$\sum_{j=d-\nu_-+1}^{d}\lambda_j^{\downarrow}(\boldsymbol{Z}) = (p_1-p_0) - \sum_{j=1}^{\nu_+}\lambda_j^{\downarrow}(\boldsymbol{Z}) \tag{17}$$

follows. Using the fact that $\|\boldsymbol{Z}\|_1 = \sum_{j=1}^{\nu_+}\lambda_j^{\downarrow}(\boldsymbol{Z}) - \sum_{j=d-\nu_-+1}^{d}\lambda_j^{\downarrow}(\boldsymbol{Z})$ alongside (17), the equalities

$$\sum_{j=1}^{\nu_+}\lambda_j^{\downarrow}(\boldsymbol{Z}) = \frac{(p_1-p_0)+\|\boldsymbol{Z}\|_1}{2} \tag{18a}$$

$$\sum_{j=d-\nu_-+1}^{d}\lambda_j^{\downarrow}(\boldsymbol{Z}) = \frac{(p_1-p_0)-\|\boldsymbol{Z}\|_1}{2} \tag{18b}$$

are obtained. The proof is completed by combining (16) with (18a) and (18b). ⊠

Theorem 1 reveals the optimal performance of QPP, but it remains to derive QPP that achieves the optimal performance. The proof of Theorem 1 provides a blueprint for deriving such QPP. Let $\boldsymbol{A}=\boldsymbol{Q}_{\boldsymbol{A}}\boldsymbol{\Lambda}_{\boldsymbol{A}}\boldsymbol{Q}_{\boldsymbol{A}}^{\dagger}$ and $\boldsymbol{B}=\boldsymbol{Q}_{\boldsymbol{B}}\boldsymbol{\Lambda}_{\boldsymbol{B}}\boldsymbol{Q}_{\boldsymbol{B}}^{\dagger}$ be spectral decompositions of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, in terms of unitary matrices $\boldsymbol{Q}_{\boldsymbol{A}}$ and $\boldsymbol{Q}_{\boldsymbol{B}}$ and diagonal matrices $\boldsymbol{\Lambda}_{\boldsymbol{A}}$ and $\boldsymbol{\Lambda}_{\boldsymbol{B}}$. The spectral theorem [33, p. 246] guarantees the existence of such decompositions since $\boldsymbol{A}$ and $\boldsymbol{B}$ are Hermitian. For any $\boldsymbol{X}\in\mathcal{U}(N_\mathrm{k}d)$ and $\boldsymbol{X}_{\boldsymbol{AB}}\triangleq\boldsymbol{Q}_{\boldsymbol{B}}^{\dagger}\boldsymbol{X}\boldsymbol{Q}_{\boldsymbol{A}}\in\mathcal{U}(N_\mathrm{k}d)$, the cyclic property of the trace gives

$$\operatorname{tr}\{\boldsymbol{X}\boldsymbol{A}\boldsymbol{X}^{\dagger}\boldsymbol{B}\} = \operatorname{tr}\{\boldsymbol{X}_{\boldsymbol{AB}}\boldsymbol{\Lambda}_{\boldsymbol{A}}\boldsymbol{X}_{\boldsymbol{AB}}^{\dagger}\boldsymbol{\Lambda}_{\boldsymbol{B}}\}\,. \tag{19}$$

Selecting $\boldsymbol{X}_{\boldsymbol{AB}}$ so that the right-hand side of (19) achieves the maximum given in (15) is straightforward: choose $\boldsymbol{X}_{\boldsymbol{AB}}$ to be a permutation matrix $\boldsymbol{P}\in\mathcal{U}(N_\mathrm{k}d)$ such that

$$\operatorname{tr}\{\boldsymbol{P}\boldsymbol{\Lambda}_{\boldsymbol{A}}\boldsymbol{P}^{\dagger}\boldsymbol{\Lambda}_{\boldsymbol{B}}\} = \operatorname{tr}\{\boldsymbol{\Lambda}_{\boldsymbol{A}}^{\downarrow}\,\boldsymbol{\Lambda}_{\boldsymbol{B}}^{\downarrow}\} = \sum_{j=1}^{N_\mathrm{k}d}\lambda_j^{\downarrow}(\boldsymbol{A})\,\lambda_j^{\downarrow}(\boldsymbol{B})\,. \tag{20}$$

Then, an optimizer of the problem in (15) is $\boldsymbol{X}^{\star}=\boldsymbol{Q}_{\boldsymbol{B}}\boldsymbol{P}\boldsymbol{Q}_{\boldsymbol{A}}^{\dagger}$. To obtain an optimizing QPP $K^{\star}(\cdot)$ for (10a), simply partition $\boldsymbol{X}^{\star}$ as in (12). The process of arriving to (20) is depicted in Fig. 1.

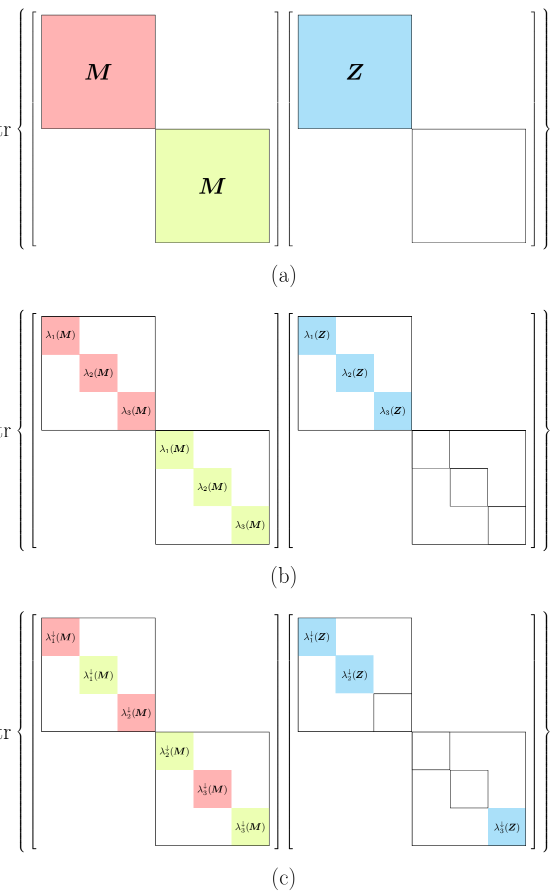

Fig. 1. Visualization of Theorem 1's proof in the case where $d=3$ and $N_\mathrm{k}=2$. The quantity $\operatorname{tr}\{\boldsymbol{X}\boldsymbol{A}\boldsymbol{X}^{\dagger}\boldsymbol{B}\}$ when $\boldsymbol{X}=\boldsymbol{I}$ is shown in (a). The quantity $\operatorname{tr}\{\boldsymbol{X}_{\boldsymbol{AB}}\boldsymbol{\Lambda}_{\boldsymbol{A}}\boldsymbol{X}_{\boldsymbol{AB}}^{\dagger}\boldsymbol{\Lambda}_{\boldsymbol{B}}\}$ when $\boldsymbol{X}_{\boldsymbol{AB}}=\boldsymbol{I}$ is shown in (b). The optimal value $\operatorname{tr}\{\boldsymbol{\Lambda}_{\boldsymbol{A}}^{\downarrow}\,\boldsymbol{\Lambda}_{\boldsymbol{B}}^{\downarrow}\}$ when $\lambda_1^{\downarrow}(\boldsymbol{Z}),\lambda_2^{\downarrow}(\boldsymbol{Z})\geqslant 0$ and $\lambda_3^{\downarrow}(\boldsymbol{Z})<0$ is shown in (c). White space is used to denote matrix elements that are decidedly zero. Notice that $\boldsymbol{\Lambda}_{\boldsymbol{B}}$ in (b) and $\boldsymbol{\Lambda}_{\boldsymbol{B}}^{\downarrow}$ in (c) have at least $(N_\mathrm{k}-1)d=3$ zero eigenvalues.

Theorem 1 provides optimal QPP for any decision rule $\mathbb{1}_{\mathcal{S}}(\cdot)$. It then remains to choose the optimal decision rule. In general, this is a difficult problem due to the nonlinear dependence of (10b) on $\boldsymbol{M}$. Fortunately, for small N, an optimal decision rule can be computed by simply enumerating (10b) (or (11) if $N_\mathrm{k}\geqslant\max\{\nu_+,\nu_-\}$) over all $\boldsymbol{M}(\mathcal{S})$ generated by subsets $\mathcal{S}$ of $\mathbb{N}_1^N$. It is interesting that the quantity $\lambda_1^{\downarrow}(\boldsymbol{M})-\lambda_1^{\uparrow}(\boldsymbol{M})$ appears in the optimal value (11). This quantity is known as the *spread* of the matrix $\boldsymbol{M}$ [35]. Hence, when $N_\mathrm{k}\geqslant\max\{\nu_+,\nu_-\}$, an optimal decision region $\mathcal{S}$ solving $\mathscr{P}_1$ is one that maximizes the spread of $\boldsymbol{M}(\mathcal{S})$.

Since $\mathcal{M}(\mathcal{S}) = \{\boldsymbol{M}(\mathcal{S}),\, \boldsymbol{I} - \boldsymbol{M}(\mathcal{S})\}$ is a POVM, this spread lies in the interval $[0, 1]$. The spread equals 1 if and only if $\lambda_1^{\downarrow}(\boldsymbol{M}(\mathcal{S})) = 1$ and $\lambda_1^{\uparrow}(\boldsymbol{M}(\mathcal{S})) = 0$. If there exists $y \in \mathbb{N}_1^N$ such that $\boldsymbol{M}_y \in \mathcal{M}$ has spread equal to 1, then an optimal decision region is simply $\mathcal{S}^\star = \{y\}$.

There is an important implication of Theorem 1. In general, it may take a mapping of Kraus rank d^2 to model a quantum channel [36, Thm 4.4.1]. However, (11) in Theorem 1 shows that at most $N_\mathrm{k} = \max\{\nu_+, \nu_-\} < d$ is needed for optimal QPP. In other words, optimal QPP does not require the most complex quantum channels. This has positive implications for performing QPP on quantum computers with limited ancilla. For qubit systems, this result shows that unitary channels are sufficient for optimal QPP.

IV. DEP Lower Bound and Comparison to the Helstrom Bound

Quantum channels are contractive under the trace norm, i.e., $\|K(\boldsymbol{Z})\|_1 \leqslant \|\boldsymbol{Z}\|_1$ for all $K(\cdot)$ and $\boldsymbol{Z}$ [36, p. 239]. Combining the contractivity of quantum channels with (6) gives the bound

$$P_\mathrm{e}^{\mathcal{M}}(\mathcal{S}; K(\cdot)) \geqslant \breve{P}_\mathrm{e} \qquad \forall\, \mathcal{S},\, K(\cdot) \tag{21}$$

and subsequently

$$\breve{P}_\mathrm{e}^{\mathcal{M}} \triangleq \underset{\mathcal{S} \subseteq \mathbb{N}_1^N,\, K(\cdot) \in \mathcal{K}(N_\mathrm{k})}{\text{minimum}} P_\mathrm{e}^{\mathcal{M}}(\mathcal{S}; K(\cdot)) \geqslant \breve{P}_\mathrm{e}\,. \tag{22}$$

That is, the minimum DEP $\breve{P}_\mathrm{e}^{\mathcal{M}}$ achievable with constrained measurement is lower bounded by the Helstrom DEP $\breve{P}_\mathrm{e}$. Note that the DEP $\breve{P}_\mathrm{e}^{\mathcal{M}}$ is achievable by construction: given fixed $\mathcal{M}$ and rank constrained QPP, $\breve{P}_\mathrm{e}^{\mathcal{M}}$ is achievable by some decision region $\mathcal{S}$ and QPP $K(\cdot)$. In the constrained measurement setting, $\breve{P}_\mathrm{e}$ is not necessarily achievable, thus $\breve{P}_\mathrm{e}^{\mathcal{M}}$ is the more informative quantity.

Let us inquire about when the DEPs $\breve{P}_\mathrm{e}^{\mathcal{M}}$ and $\breve{P}_\mathrm{e}$ coincide, in which case optimal QPP alongside an optimal decision rule allows one to achieve the Helstrom bound despite the fixed measurement. The following theorem gives necessary and sufficient conditions for the two DEPs to coincide.

Theorem 2: Recall from Remark 1 that $\boldsymbol{Z}$ is indefinite, i.e., $\nu_+ > 0$ and $\nu_- > 0$. The following statements hold.

- If $N_\mathrm{k} \geqslant \max\{\nu_+, \nu_-\}$, the equality $\breve{P}_\mathrm{e}^{\mathcal{M}} = \breve{P}_\mathrm{e}$ in (22) holds if and only if there exists a decision region $\mathcal{S} \subseteq \mathbb{N}_1^N$ satisfying the following *equivalent* conditions:
  1) the spread of $\boldsymbol{M}(\mathcal{S})$ equals 1;
  2) $\lambda_1^{\downarrow}(\boldsymbol{M}(\mathcal{S})) = 1$ and $\lambda_1^{\uparrow}(\boldsymbol{M}(\mathcal{S})) = 0$.
- If $N_\mathrm{k} < \max\{\nu_+, \nu_-\}$, the equality $\breve{P}_\mathrm{e}^{\mathcal{M}} = \breve{P}_\mathrm{e}$ in (22) holds if and only if there exists a decision region $\mathcal{S} \subseteq \mathbb{N}_1^N$ satisfying *both* of the following conditions:
  1) $\boldsymbol{M}(\mathcal{S})$ has at least ν_+/N_k eigenvalues equal to 1;
  2) $\boldsymbol{M}(\mathcal{S})$ has at least ν_-/N_k eigenvalues equal to 0.

□

Proof: Consider first the case where $N_\mathrm{k} \geqslant \max\{\nu_+, \nu_-\}$. From (8) and (11), the optimal DEP is given by

$$\breve{P}_\mathrm{e}^{\mathcal{M}} = p_1 - \underset{\mathcal{S} \subseteq \mathbb{N}_1^N}{\text{maximum}}\, \frac{\lambda_1^{\downarrow}(\boldsymbol{M}(\mathcal{S})) - \lambda_1^{\uparrow}(\boldsymbol{M}(\mathcal{S}))}{2} \times \big(\|\boldsymbol{Z}\|_1 + (p_1 - p_0)\big). \tag{23}$$

Note that the eigenvalues of any POVM element, such as $\boldsymbol{M}(\mathcal{S})$, lie in the interval $[0, 1]$. Hence,

$$\lambda_1^{\downarrow}(\boldsymbol{M}(\mathcal{S})) - \lambda_1^{\uparrow}(\boldsymbol{M}(\mathcal{S})) \leqslant 1\,. \tag{24}$$

Since $\|\boldsymbol{Z}\|_1 + (p_1 - p_0) = \sum_{j=1}^d |\lambda_j^{\downarrow}(\boldsymbol{Z})| + \lambda_j^{\downarrow}(\boldsymbol{Z}) > 0$, it follows that $\breve{P}_\mathrm{e}^{\mathcal{M}}$ is maximized if and only if $\lambda_1^{\downarrow}(\boldsymbol{M}(\mathcal{S})) - \lambda_1^{\uparrow}(\boldsymbol{M}(\mathcal{S})) = 1$. Equivalently, $\breve{P}_\mathrm{e}^{\mathcal{M}}$ is maximized if and only if both $\lambda_1^{\downarrow}(\boldsymbol{M}(\mathcal{S})) = 1$ and $\lambda_1^{\uparrow}(\boldsymbol{M}(\mathcal{S})) = 0$. If this is the case, using the fact that $p_0 + p_1 = 1$ in (23) reveals $\breve{P}_\mathrm{e}^{\mathcal{M}} = \breve{P}_\mathrm{e}$.

Consider now the case where $N_\mathrm{k} < \max\{\nu_+, \nu_-\}$. Recall that the eigenvalues of $\boldsymbol{I}_{N_\mathrm{k}} \otimes \boldsymbol{M}(\mathcal{S})$ are exactly those of $\boldsymbol{M}(\mathcal{S})$ with multiplicity N_k. The right-hand side of (10b) is maximized if and only if $\boldsymbol{M}(\mathcal{S})$ has at least ν_+/N_k eigenvalues equal to 1 and $\boldsymbol{M}(\mathcal{S})$ has at least ν_-/N_k eigenvalues equal to 0. If there exists $\mathcal{S}$ such that $\boldsymbol{M}(\mathcal{S})$'s eigenvalues satisfy these two conditions, then $\breve{P}_\mathrm{e}^{\mathcal{M}}$ is

$$\begin{aligned}
\breve{P}_\mathrm{e}^{\mathcal{M}} &= p_1 - \sum_{j=1}^{\nu_+} \lambda_j^{\downarrow}(\boldsymbol{Z}) && \text{(25a)}\\
&= p_1 - \frac{(p_1 - p_0) + \|\boldsymbol{Z}\|_1}{2} && \text{(25b)}\\
&= \breve{P}_\mathrm{e} && \text{(25c)}
\end{aligned}$$

where (25b) follows from (18a) and (25c) follows from $p_0 + p_1 = 1$. This completes the proof. ⊠

V. Case Study: Quantum Computer-Enabled Receiver for Optical BPSK with Coherent States

This section will use the theory developed in this paper to quantify the optimal DEP achievable by a quantum computer-enabled receiver for optical BPSK communications with coherent states, such as that proposed in [31]. Quantum transduction [37] is used to transfer the information encoded in the received optical state to a superconducting qubit. For the details of this process, the reader is referred to [31].

The transduction occurs in two steps: the optical state is transduced to a microwave cavity and then the state of the microwave cavity is transduced to a superconducting qubit. After the first step, given that the coherent state $|\pm\beta\rangle$ is received, $\beta \in \mathbb{C}$, the cavity is in the Gaussian state

$$\boldsymbol{\Xi}^{\mathrm{M}}(\bar{n}_\mathrm{tr}, \pm\sqrt{\eta_\mathrm{tr}}\beta) = \frac{1}{\pi\, \bar{n}_\mathrm{tr}} \int_{\mathbb{C}} e^{-\frac{|\alpha \mp \sqrt{\eta_\mathrm{tr}}\beta|^2}{\bar{n}_\mathrm{tr}}} |\alpha\rangle\langle\alpha|\, \mathrm{d}\alpha \tag{26}$$

where $|\alpha\rangle$ is a coherent state, $\eta_\mathrm{tr} \geqslant 0$ is the loss parameter, and $\bar{n}_\mathrm{tr} \geqslant 0$ is the "heating parameter" (number of thermal photons accumulated during the transduction process). The $\pm\sqrt{\eta_\mathrm{tr}}\,\beta$ in (26) carries the information about which BPSK symbol was sent. The state $\boldsymbol{\Xi}^{\mathrm{M}}(\bar{n}_\mathrm{tr}, \pm\sqrt{\eta_\mathrm{tr}}\,\beta)$ of the microwave cavity is then transduced to the superconducting qubit via the

TABLE I
OPTIMAL DEP ACHIEVABLE WITH QPP OF KRAUS RANK CONSTRAINT N_k AND MEASUREMENT EFFICIENCY η.

Copies of the unknown state: $m = 4$
Helstrom bound: $\breve{P}_\mathrm{e} = 0.0286$

	$N_\mathrm{k} = 1$	$N_\mathrm{k} = 2$	$N_\mathrm{k} = 3$	$N_\mathrm{k} = 4$
$\eta = 0.1$	0.0747	0.0629	0.0550	0.0550
$\eta = 0.2$	0.1529	0.1371	0.1267	0.1267
$\eta = 0.3$	0.2552	0.2414	0.2323	0.2323
$\eta = 0.4$	0.3736	0.3657	0.3605	0.3605

Copies of the unknown state: $m = 3$
Helstrom bound: $\breve{P}_\mathrm{e} = 0.0267$

	$N_\mathrm{k} = 1$	$N_\mathrm{k} = 2$	$N_\mathrm{k} = 3$	$N_\mathrm{k} = 4$
$\eta = 0.1$	0.0766	0.0687	0.0608	0.0532
$\eta = 0.2$	0.1563	0.1458	0.1353	0.1252
$\eta = 0.3$	0.2584	0.2492	0.2401	0.2312
$\eta = 0.4$	0.3755	0.3702	0.3650	0.3599

Copies of the unknown state: $m = 2$
Helstrom bound: $\breve{P}_\mathrm{e} = 0.0784$

	$N_\mathrm{k} = 1$	$N_\mathrm{k} = 2$	$N_\mathrm{k} = 3$	$N_\mathrm{k} = 4$
$\eta = 0.1$	0.1627	0.1627	0.1627	0.1627
$\eta = 0.2$	0.2470	0.2470	0.2470	0.2470
$\eta = 0.3$	0.3314	0.3314	0.3314	0.3314
$\eta = 0.4$	0.4157	0.4157	0.4157	0.4157

Jaynes-Cummings interaction. In the interaction picture, the Hamiltonian is $\boldsymbol{H} = \hbar\chi(\boldsymbol{A}\otimes\boldsymbol{\Sigma}_+ + \boldsymbol{A}^\dagger\otimes\boldsymbol{\Sigma}_-)$ where $\hbar$ is the reduced Planck constant, $\chi \geqslant 0$ is the coupling between the qubit and field, $\boldsymbol{A}$ is the annihilation operator of the field, and $\boldsymbol{\Sigma}_+$ (resp. $\boldsymbol{\Sigma}_-$) is the raising (resp. lowering) operator of the qubit. Denoting the mean photon number of the microwave cavity by N, consider the case where $\bar{n}_\mathrm{tr} = 1.8$, $\eta_\mathrm{tr} = 0.924$, $\chi t = \pi/4\sqrt{N}$, and $\beta = 3$ [31].

After transduction, the state $\boldsymbol{\Upsilon}$ of the qubit takes on one of two values, $\boldsymbol{\Upsilon}_0$ or $\boldsymbol{\Upsilon}_1$, depending on which symbol $|\pm\beta\rangle$ was sent by the transmitter. Take the prior probabilities to be equally likely: $p_0 = p_1 = 1/2$. In the case of m-fold repetition encoding, we can transduce m independently received symbols to m different qubits. The goal of the receiver is to infer $\boldsymbol{\Xi}$, taking on values $\boldsymbol{\Xi}_0 = \boldsymbol{\Upsilon}_0^{\otimes m}$ and $\boldsymbol{\Xi}_1 = \boldsymbol{\Upsilon}_1^{\otimes m}$. Suppose that imperfect measurements, characterized by the binary POVM $\mathcal{M} = \{\boldsymbol{M}, \boldsymbol{I} - \boldsymbol{M}\}$ with

$$\boldsymbol{M} = (1-\eta)\,|0\rangle\langle 0| + \eta\,|1\rangle\langle 1| \tag{27}$$

can be performed independently on each qubit. In (27), $\eta \in (0, 0.5)$ is the efficiency of the measurement. In the limit of $\eta \to 0$, (27) represents a projective measurement onto the computational basis of each qubit. Finally, owing to the fact that the qubits reside in a quantum computer, they can be coupled to an N_k-dimensional ancilla. By the Stinespring dilation theorem [36, p. 172], preparing the ancilla in a pure state followed by performing composite unitary operations allows one to achieve arbitrary QPP of Kraus rank constraint N_k on $\boldsymbol{\Xi}$.

Table I summarizes the minimum DEP $\breve{P}_\mathrm{e}^{\mathcal{M}}$ and the Helstrom bound $\breve{P}_\mathrm{e}$ for various measurement efficiencies η, ancilla dimension N_k, and block length m. The Helstrom bound is at least an order of magnitude off from the minimum achievable DEP in many cases. Fig. 2 plots the ratio between the Helstrom bound and the minimum DEP as a function of measurement efficiency η in the case where $N_\mathrm{k} = 2$. Notice that, even in the case where the measurement efficiency is around 95%, this ratio is significantly less than 1.

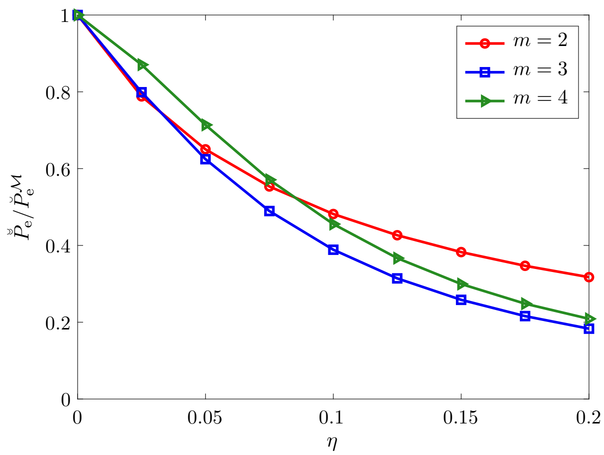

Fig. 2. Ratio between the Helstrom bound and the optimal DEP, when $N_\mathrm{k} = 2$, plotted as a function of the measurement efficiency η. The lower this ratio, the looser the Helstrom bound is in the fixed measurement setting.

VI. CONCLUSION

This paper explored quantum state discrimination in the setting where a fixed measurement apparatus is available. The employment of optimal QPP was advocated and the fundamental limits of quantum state discrimination—as well as optimal QPP to achieve these limits—were derived in closed form. The derived limits supplant the Helstrom bound in the scenarios of interest. The results derived in this paper can play a key role in the design of quantum systems relying on state discrimination.

REFERENCES

[1] C. W. Helstrom, "Detection theory and quantum mechanics," *Info. and Cont.*, vol. 10, no. 3, pp. 254–291, 1967.
[2] C. Helstrom, *Quantum detection and estimation theory*. New York, USA: Academic Press, 1976.
[3] A. Giani, M. Z. Win, and A. Conti, "Quantum sensing and communication via non-Gaussian states," *IEEE J. Sel. Areas Inf. Theory*, pp. 1–16, 2025, early access.
[4] G. Y. Tham, R. Nair, and M. Gu, "Quantum limits of covert target detection," *Phys. Rev. Lett.*, vol. 133, no. 11, p. 110801, 2024.
[5] A. Karsa, A. Fletcher, G. Spedalieri, and S. Pirandola, "Quantum illumination and quantum radar: A brief overview," *Rep. prog. phys.*, vol. 87, no. 9, p. 094001, 2024.
[6] S. Guerrini, M. Z. Win, M. Chiani, and A. Conti, "Quantum discrimination of noisy photon-added coherent states," *IEEE J. Sel. Areas Inf. Theory*, vol. 1, no. 2, pp. 469–479, Aug. 2020, special issue on *Quantum Information Science*.
[7] M. Clouâtré, S. Marano, P. L. Falb, and M. Z. Win, "Admissible optimal control for parameter estimation in quantum systems," *IEEE Contr. Syst. Lett.*, vol. 8, pp. 2283–2288, Sep. 2024.
[8] H. Ma, D. Dong, I. R. Petersen, C.-J. Huang, and G.-Y. Xiang, "Neural networks for quantum state tomography with constrained measurements," *Quantum Inf. Process.*, vol. 23, no. 9, p. 317, 2024.

[9] Y. Liu and J. Acharya, "Quantum state testing with restricted measurements," *arXiv preprint arXiv:2408.17439*, 2024.
[10] G. Sorelli, N. Treps, F. Grosshans, and F. Boust, "Detecting a target with quantum entanglement," *IEEE Aerosp. Electron. Syst. Mag.*, vol. 37, no. 5, pp. 68–90, 2021.
[11] R. Malaney, "The quantum car," *IEEE Wireless Commun. Lett.*, vol. 5, no. 6, pp. 624–627, 2016.
[12] S. Olivares, S. Cialdi, F. Castelli, and M. G. Paris, "Homodyne detection as a near-optimum receiver for phase-shift-keyed binary communication in the presence of phase diffusion," *Phys. Rev. A*, vol. 87, no. 5, p. 050303, 2013.
[13] P. Hausladen and W. K. Wootters, "A 'pretty good' measurement for distinguishing quantum states," *J. Mod. Opt.*, vol. 41, no. 12, pp. 2385–2390, 1994.
[14] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, "Quantum nonlocality without entanglement," *Phys. Rev. A*, vol. 59, no. 2, p. 1070, 1999.
[15] W. Matthews, S. Wehner, and A. Winter, "Distinguishability of quantum states under restricted families of measurements with an application to quantum data hiding," *Comm. Math. Phys.*, vol. 291, pp. 813–843, 2009.
[16] C. Lancien and A. Winter, "Distinguishing multi-partite states by local measurements," *Comm. Math. Phys.*, vol. 323, pp. 555–573, 2013.
[17] F. G. Brandao, A. W. Harrow, J. R. Lee, and Y. Peres, "Adversarial hypothesis testing and a quantum Stein's lemma for restricted measurements," *IEEE Trans. Inf. Theory*, vol. 66, no. 8, pp. 5037–5054, 2020.
[18] W. Matthews, M. Piani, and J. Watrous, "Entanglement in channel discrimination with restricted measurements," *Phys. Rev. A*, vol. 82, no. 3, p. 032302, 2010.
[19] J. Bae and L.-C. Kwek, "Quantum state discrimination and its applications," *J. Phys. A: Math. Theor.*, vol. 48, no. 8, p. 083001, 2015.
[20] R. Li, P. Narang, M. Aelmans, P. Mueller, and G. Long, "Quantum communications and networking: Series 1," *IEEE Netw.*, vol. 38, no. 1, pp. 77–78, 2024.
[21] L. Hanzo, Z. Babar, Z. Cai, D. Chandra, I. B. Djordjevic, B. Koczor, S. X. Ng, M. Razavi, and O. Simeone, "Quantum information processing, sensing, and communications: Their myths, realities, and futures," *Proc. IEEE*, 2025, early access.
[22] M. S. Winnel, Z. Wang, R. Malaney, R. Aguinaldo, J. Green, and T. C. Ralph, "Classical-quantum dual encoding for laser communications in space," *New J. Phys.*, vol. 26, no. 3, p. 033012, 2024.
[23] Z. Liu, S. Marano, and M. Z. Win, "Establishing high-fidelity entanglement in quantum repeater chains," *IEEE J. Sel. Areas Commun.*, vol. 42, no. 7, pp. 1763–1778, Jul. 2024, special issue on *The Quantum Internet: Principles, Protocols, and Architectures*.
[24] L. Yang, Y. Zhao, H. Xu, L. Huang, and C. Qiao, "Dynamic entanglement routing based on stream processing for quantum networks," *IEEE/ACM Trans. Netw.*, vol. 33, no. 1, pp. 382–397, 2024.
[25] J. Illiano, M. Caleffi, A. Manzalini, and A. S. Cacciapuoti, "Quantum Internet protocol stack: A comprehensive survey," *Computer Networks*, vol. 213, p. 109092, Aug. 2022.
[26] A. S. Cacciapuoti, M. Caleffi, R. Van Meter, and L. Hanzo, "When entanglement meets classical communications: Quantum teleportation for the Quantum Internet," *IEEE Trans. Commun.*, vol. 68, no. 6, pp. 3808–3833, Jun. 2020.
[27] M. Chiani, A. Conti, and M. Z. Win, "Piggybacking on quantum streams," *Phys. Rev. A*, vol. 102, p. 012410, Jul. 2020.
[28] A. S. Cacciapuoti, M. Caleffi, F. Tafuri, F. S. Cataliotti, S. Gherardini, and G. Bianchi, "Quantum Internet: Networking challenges in distributed quantum computing," *IEEE Netw.*, vol. 34, no. 1, pp. 137–143, January/February 2019.
[29] A. S. Fletcher, P. W. Shor, and M. Z. Win, "Structured near-optimal channel-adapted quantum error correction," *Phys. Rev. A*, vol. 77, p. 012320, Jan. 2008.
[30] N. Gisin and R. Thew, "Quantum communication," *Nat. Photonics*, vol. 1, no. 3, pp. 165–171, 2007.
[31] J. Crossman, S. Dimitroff, L. Cincio, and M. Sarovar, "Quantum computer-enabled receivers for optical communication," *Quantum Sci. Tech.*, vol. 9, no. 4, p. 045005, 2024.
[32] P. J. Bickel and K. Doksum, *Mathematical Statistics: Basic Ideas and Selected Topics*, 1st ed. Oakland, California: Holden-Day, Inc., 1977.
[33] S. Axler, *Linear algebra done right*, 4th ed. Springer Nature, 2024.
[34] R. W. Brockett, "Least squares matching problems," *Linear Algebra Appl.*, vol. 122, pp. 761–777, 1989.
[35] L. Mirsky, "The spread of a matrix," *Mathematika*, vol. 3, no. 2, pp. 127–130, 1956.
[36] M. M. Wilde, *Quantum information theory*, 2nd ed. Cambridge university press, 2017.
[37] N. Lauk, N. Sinclair, S. Barzanjeh, J. P. Covey, M. Saffman, M. Spiropulu, and C. Simon, "Perspectives on quantum transduction," *Quantum Sci. Tech.*, vol. 5, no. 2, p. 020501, 2020.