Photon-varied quantum states: Unified characterization

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This paper introduces photon-varied quantum states (PVQSs), which generalize the nonclassical states obtained via photon addition or subtraction operations. We provide a unified characterization of PVQSs in terms of characteristic function, quasiprobability distribution, Fock representation, and Mandel Q parameter. In the special case of photon-varied Gaussian states (PVGSs), the characteristic functions and the quasiprobability distributions are found to be in a simple canonical product structure. Necessary and sufficient conditions for the negativity of the quasiprobability distributions are also obtained for PVGSs. The unified characterization enables the design and analysis of quantum systems that exploit the non-Gaussian properties of PVQSs.

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I. INTRODUCTION

Nonclassical states are a key enabler for quantum communications [1–4], quantum sensing and metrology [5–11], quantum computation [12–14], and quantum cryptography [15–18] in both the optical [19–22] and microwave [23–26] domains. In particular, Gaussian states (e.g., squeezed states) have been considered extensively in quantum information theory for providing nonclassicality in continuous variables systems [27–33]. However, Gaussian states lack some desirable properties (e.g., Wigner function negativity) [30] for quantum supremacy in various applications, including quantum sensing and quantum computing [10,34]. Therefore, it is important to identify and characterize new classes of non-Gaussian states that offer performance gain, yet are easy to prepare, in quantum systems and networks.

Photon-added quantum states (PAQSs) [35–38] and photon-subtracted quantum states (PSQSs) [39–43] are two important classes of non-Gaussian states that exhibit nonclassical behaviors [44–49]. The non-Gaussian quantum states obtained by performing photon-addition or photon-subtraction operations on a Gaussian state are called photon-added Gaussian states (PAGSs) and photon-subtracted Gaussian states (PSGSs), respectively. The benefits of PAGSs and PSGSs have been shown for several applications, including quantum communications [50–52], quantum key distribution [53–55], and quantum sensing [56–58]. While significant progress has been made over the last three decades [4,35–43], a complete and unified characterization of photon-added and photon-subtracted states (in terms of characteristic functions, quasiprobability distributions, Fock representation, and Mandel Q parameter) is missing.

The goal of this paper is to characterize the classes of PAQSs and PSQSs in a unified framework. Hereafter, we refer to these classes of quantum states as photon-varied quantum states (PVQSs). We show that photon-varied Gaussian states (PVGSs) have a simple canonical structure and exhibit a nonclassical behavior, including negative quasiprobability distributions and a sub-Poissonian photon number distribution (i.e., negative Mandel Q parameter [59]). This paper develops a framework for a unified characterization of PVQSs (see

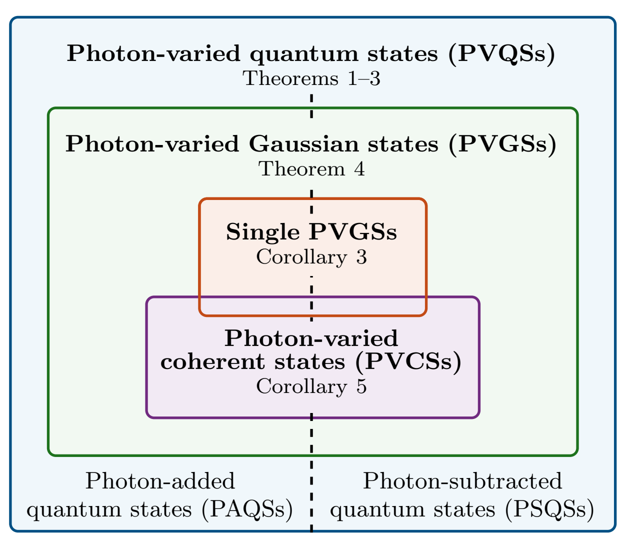

FIG. 1. Schematic representation of the different classes of photon-varied states examined in this paper.

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Fig. 1). The key contributions of this paper can be summarized as follows:

(i) we characterize PVQSs in terms of characteristic function, quasiprobability distribution, Fock representation, and Mandel Q parameter; and

(ii) we provide the unified characterization for PVGSs in a simple canonical product structure and quantify their nonclassicality.

The characterization of PVQSs enables the design of quantum states with desirable nonclassical properties.

The remaining sections are organized as follows. Section II establishes a framework for the characterization of PVQSs. Section III characterizes PVGSs in a canonical product structure. Final remarks are given in Sec. IV.

Notations. Operators are denoted by bold uppercase letters. The sets of complex numbers and of positive integers are denoted by $\mathbb{C}$ and $\mathbb{N}$, respectively. For $n \in \mathbb{Z}$: $\overline{n}| \equiv +$ for $n \geqslant 0$, and $\overline{n}| \equiv -$ for $n < 0$. For $z \in \mathbb{C}$: $|z|$ and $\arg(z)$ denote the absolute value and the argument, respectively; z_r and z_i denote the real part and the imaginary part, respectively; z^* is the complex conjugate; $\check{z} = [z \;\; z^*]^T$ is the augmented vector associated with z, and $\imath = \sqrt{-1}$. For $z \in \mathbb{C}$, the Wirtinger derivatives are defined as $\frac{\partial}{\partial z} = \frac{1}{2}\big(\frac{\partial}{\partial z_\mathrm{r}} - \imath\frac{\partial}{\partial z_\mathrm{i}}\big)$ and $\frac{\partial}{\partial z^*} = \frac{1}{2}\big(\frac{\partial}{\partial z_\mathrm{r}} + \imath\frac{\partial}{\partial z_\mathrm{i}}\big)$. The trace and the adjoint of an operator are denoted by $\mathrm{tr}\{\cdot\}$ and $(\cdot)^\dagger$, respectively. The annihilation, the creation, and the identity operators are denoted by $\boldsymbol{A}$, $\boldsymbol{A}^\dagger$, and $\boldsymbol{I}$, respectively. The displacement operator with parameter $\mu \in \mathbb{C}$ is $\boldsymbol{D}_\mu = \exp\{\mu\boldsymbol{A}^\dagger - \mu^*\boldsymbol{A}\}$. The rotation operator with parameter $\phi \in \mathbb{R}$ is $\boldsymbol{R}_\phi = \exp\{-\imath\phi\boldsymbol{A}^\dagger\boldsymbol{A}\}$. The squeezing operator with parameter $r \in \mathbb{R}$ is $\boldsymbol{S}_r = \exp\{\frac{1}{2}r(\boldsymbol{A}^\dagger)^2 - \frac{1}{2}r\boldsymbol{A}^2\}$. For two operators $\boldsymbol{X}$ and $\boldsymbol{Y}$, the anticommutator and the commutator are denoted by $[\![\boldsymbol{X}, \boldsymbol{Y}]\!]_\pm = \boldsymbol{XY} \pm \boldsymbol{YX}$ with $+$ and $-$, respectively. For a quantum state $\boldsymbol{\Xi}$, the expectation value of an observable $\boldsymbol{A}$ is $\langle \boldsymbol{A}\rangle = \mathrm{tr}\{\boldsymbol{\Xi}\boldsymbol{A}\}$. Notation $\boldsymbol{M}^+$ indicates the Moore–Penrose pseudoinverse of a matrix $\boldsymbol{M}$ [60].

II. PHOTON-VARIED QUANTUM STATES

Consider a single bosonic mode described by the quadrature operators $\boldsymbol{Q}$ and $\boldsymbol{P}$ satisfying the canonical commutation relation $[\![\boldsymbol{Q}, \boldsymbol{P}]\!]_- = \imath\boldsymbol{I}$, and let $\boldsymbol{A} = (\boldsymbol{Q} + \imath\boldsymbol{P})/\sqrt{2}$ and $\boldsymbol{A}^\dagger = (\boldsymbol{Q} - \imath\boldsymbol{P})/\sqrt{2}$ [61]. Let $\boldsymbol{\Xi}$ be the density operator representing a state of the single bosonic mode. The PAQS associated with $\boldsymbol{\Xi}$ is defined as

$$\boldsymbol{\Xi}_+^{(k)} = \frac{(\boldsymbol{A}^\dagger)^k\boldsymbol{\Xi}\boldsymbol{A}^k}{N_+^{(k)}}, \tag{1}$$

where $k \in \mathbb{N}$ is the number of addition operations, and $N_+^{(k)} = \mathrm{tr}\{(\boldsymbol{A}^\dagger)^k\boldsymbol{\Xi}\boldsymbol{A}^k\}$ is the normalization constant. Analogously, the PSQS associated with $\boldsymbol{\Xi}$ is defined as

$$\boldsymbol{\Xi}_-^{(k)} = \frac{\boldsymbol{A}^k\boldsymbol{\Xi}(\boldsymbol{A}^\dagger)^k}{N_-^{(k)}}, \tag{2}$$

where $k \in \mathbb{N}$ is the number of subtraction operations, and $N_-^{(k)} = \mathrm{tr}\{\boldsymbol{A}^k\boldsymbol{\Xi}(\boldsymbol{A}^\dagger)^k\}$ is the normalization constant.

For notational convenience, we introduce the notation $\boldsymbol{\Xi}_{\overline{t}|}^{(k)}$ and $N_{\overline{t}|}^{(k)}$ for unifying the characterization of PAQSs ($t = +1$) and PSQSs ($t = -1$), obtained from the initial state $\boldsymbol{\Xi}$. Note that the PVQS $\boldsymbol{\Xi}_{\overline{t}|}^{(k)}$ has the same rotation symmetry as the initial state $\boldsymbol{\Xi}$, i.e., a rotation of the initial state $\boldsymbol{\Xi}$ produces a corresponding rotation to $\boldsymbol{\Xi}_{\overline{t}|}^{(k)}$.

A. Characteristic function

For a quantum state $\boldsymbol{\Xi}$, the s-ordered characteristic function $\chi(\xi, s)$ is defined by [62]

$$\chi(\xi, s) = \exp\left\{\frac{s}{2}|\xi|^2\right\}\mathrm{tr}\{\boldsymbol{\Xi}\boldsymbol{D}_\xi\}. \tag{3}$$

Note that the characteristic function can be used to determine the normalization constant $N_{\overline{t}|}^{(k)}$ as [62, Eq. (6.26)]

$$N_{\overline{t}|}^{(k)} = \mathrm{tr}\{\boldsymbol{\Xi}\{(\boldsymbol{A}^\dagger)^k\boldsymbol{A}^k\}_{-t}\} = \frac{\partial^{2k}}{\partial\xi^k\partial(-\xi^*)^k}\,\chi(\xi, -t)\Big|_{\xi=0},$$

where $\{(\boldsymbol{A}^\dagger)^k\boldsymbol{A}^k\}_s$ denotes the s-ordered product of $(\boldsymbol{A}^\dagger)^k$ and $\boldsymbol{A}^k$, with $s \in \mathbb{C}$, as defined in [63]. Recall that the normal, antinormal, and symmetrically ordered products are obtained with $s = 1$, $s = -1$, and $s = 0$, respectively. Note also that the use of definition (3) for determining the characteristic function of a PVQS does not reveal the functional relationship between the PVQS and the corresponding initial state.

The following theorem relates the characteristic function of a PVQS $\boldsymbol{\Xi}_{\overline{t}|}^{(k)}$ to that of the initial state $\boldsymbol{\Xi}$.

Theorem 1 (Characteristic function of a PVQS). Let $\chi(\xi, s)$ and $\chi_{\overline{t}|}^{(k)}(\xi, s)$ be the s-ordered characteristic function associated with $\boldsymbol{\Xi}$ and $\boldsymbol{\Xi}_{\overline{t}|}^{(k)}$, respectively. The relation between the two characteristic functions is given by

$$\chi_{\overline{t}|}^{(k)}(\xi, s) = \frac{(-1)^k}{N_{\overline{t}|}^{(k)}}\,e^{\frac{s+t}{2}|\xi|^2}\,\frac{\partial^{2k}}{\partial\xi^k\partial\xi^{*k}}\,\chi(\xi, s)\,e^{-\frac{s+t}{2}|\xi|^2}. \tag{4}$$

Proof. See Appendix A. ⊠

B. Quasiprobability distribution

For a quantum state $\boldsymbol{\Xi}$, the s-ordered quasiprobability distribution $W(\alpha, s)$ is defined by [62]

$$W(\alpha, s) = \frac{1}{\pi^2}\int_{\mathbb{R}^2}\chi(\xi, s)\,e^{\alpha\xi^* - \alpha^*\xi}\,d^2\xi, \tag{5}$$

where $d^2\xi = d\xi_\mathrm{r}d\xi_\mathrm{i}$. Recall that the Wigner W function, the Glauber-Sudarshan P function, and the Husimi Q function are obtained with $s = 0$, $s = 1$, and $s = -1$, respectively [27–31].

The following theorem relates the s-ordered quasiprobability distribution of a PVQS $\boldsymbol{\Xi}_{\overline{t}|}^{(k)}$ to that of the initial state $\boldsymbol{\Xi}$.

Theorem 2 (Quasiprobability distribution of a PVQS). Let $W(\alpha, s)$ and $W_{\overline{t}|}^{(k)}(\alpha, s)$ be the s-ordered quasiprobability distribution associated with $\boldsymbol{\Xi}$ and $\boldsymbol{\Xi}_{\overline{t}|}^{(k)}$, respectively. For $s \neq -t$, the relation between the two s-ordered quasiprobability distributions is given by

$$W_{\overline{t}|}^{(k)}(\alpha, s) = \frac{(s+t)^{2k}}{4^k N_{\overline{t}|}^{(k)}}\,e^{\frac{2|\alpha|^2}{s+t}}\,\frac{\partial^{2k}}{\partial\alpha^k\partial\alpha^{*k}}\,W(\alpha, s)\,e^{-\frac{2|\alpha|^2}{s+t}}. \tag{6}$$

For $s = -t$, the relation is found to be

$$W_{\bar{t}|}^{(k)}(\alpha, -t) = \frac{|\alpha|^{2k}}{N_{\bar{t}|}^{(k)}} W(\alpha, -t). \tag{7}$$

Proof. See Appendix B. ☒

Remark. Theorems 1 and 2 establish simple and parallel differential relations between PVQS $\Xi_{\bar{t}|}^{(k)}$ and initial state Ξ in terms of characteristic function and quasiprobability distribution, respectively.

C. Fock representation

For a quantum state Ξ, the representation in the Fock basis $\{|n\rangle\}_{n\in\mathbb{N}}$ is given by

$$\Xi = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\langle n|\Xi|m\rangle|n\rangle\langle m|. \tag{8}$$

The following theorem relates the Fock representation of a PVQS $\Xi_{\bar{t}|}^{(k)}$ and that of the initial state Ξ.

Theorem 3 (Fock representation of a PVQS). The relation between the Fock representation of $\Xi_{\bar{t}|}^{(k)}$ and that of Ξ is found to be

$$\langle n|\Xi_{\bar{t}|}^{(k)}|m\rangle = \frac{1}{N_{\bar{t}|}^{(k)}}\begin{cases}\zeta_{n,m}^{(k)}\langle n+k|\Xi|m+k\rangle & \text{for } t=-1\\ c_{n,m}(\Xi) & \text{for } t=+1,\end{cases} \tag{9}$$

where

$$\zeta_{n,m}^{(k)} = \sqrt{\frac{(n+k)!(m+k)!}{n!m!}}$$

$$c_{n,m}(\Xi) = \begin{cases}\zeta_{n-k,m-k}^{(k)}\langle n-k|\Xi|m-k\rangle & \\ & \text{for both } n, m \geqslant k\\ 0 & \text{otherwise.}\end{cases}$$

Proof. See Appendix C. ☒

D. Nonclassical properties

For a quantum state Ξ, the Mandel Q parameter is an indicator of its nonclassicality, which quantifies the sub-Poissonian behavior of the photon number statistic, defined as [59]

$$M_{\mathrm{Q}} = \frac{\langle(\boldsymbol{A}^{\dagger})^2\boldsymbol{A}^2\rangle - \langle\boldsymbol{A}^{\dagger}\boldsymbol{A}\rangle^2}{\langle\boldsymbol{A}^{\dagger}\boldsymbol{A}\rangle}. \tag{10}$$

In particular, by using the antinormal order form [63] of $(\boldsymbol{A}^{\dagger})^n\boldsymbol{A}^n$, we obtain

$$\langle(\boldsymbol{A}^{\dagger})^n\boldsymbol{A}^n\rangle_{\bar{t}|} = \begin{cases}\frac{N_-^{(k+n)}}{N_-^{(k)}} & \text{for } t=-1\\ \frac{(-1)^n n!}{N_+^{(k)}}\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{j!}N_+^{(k+j)} & \text{for } t=+1.\end{cases} \tag{11}$$

Note that (11) is general in n. The Mandel M_{Q} parameter for a PAQS $\Xi_+^{(k)}$ and a PSQS $\Xi_-^{(k)}$ is obtained by applying (11) with $n = 2$ in (10) as given by

$$M_{\mathrm{Q}\,\bar{t}|}^{(k)} = \begin{cases}\frac{N_-^{(k+2)}}{N_-^{(k+1)}} - \frac{N_-^{(k+1)}}{N_-^{(k)}} & \text{for } t=-1\\ \frac{N_+^{(k+2)}-2N_+^{(k)}}{N_+^{(k+1)}-N_+^{(k)}} - \frac{N_+^{(k+1)}}{N_+^{(k)}} - 3 & \text{for } t=+1.\end{cases}$$

III. PHOTON-VARIED GAUSSIAN STATES

This section shows how to utilize the results of Sec. II to characterize the quantum states obtained by applying a photon-variation operation on a Gaussian state.

A. Preliminaries

1. Single-mode Gaussian states

A single-mode Gaussian state is a quantum state with a Gaussian Wigner function in the $\mathbb{R}^2$ phase space spanned by the eigenvalues of $\boldsymbol{Q}$ and $\boldsymbol{P}$ [27–31], i.e.,

$$\mathring{W}_{\mathrm{G}}(\boldsymbol{x}) = \frac{1}{\pi\sqrt{\det\boldsymbol{V}}}\exp\left\{-\frac{1}{2}(\boldsymbol{x}-\bar{\boldsymbol{x}})^{\mathrm{T}}\boldsymbol{V}^{-1}(\boldsymbol{x}-\bar{\boldsymbol{x}})\right\}, \tag{12}$$

where $\boldsymbol{x} = [q\ p]^{\mathrm{T}} \in \mathbb{R}^2$ is the vector of eigenvalues of $\boldsymbol{Q}$ and $\boldsymbol{P}$, $\bar{\boldsymbol{x}} = [\bar{q}\ \bar{p}]^{\mathrm{T}} \in \mathbb{R}^2$ is the mean value, and $\boldsymbol{V}$ is the covariance matrix with entries $[\boldsymbol{V}]_{i,j} = 2^{-1}\langle[\![\boldsymbol{X}_i - \langle\boldsymbol{X}_i\rangle, \boldsymbol{X}_j - \langle\boldsymbol{X}_j\rangle]\!]_+\rangle$, and $\boldsymbol{X} = [\boldsymbol{Q}\,\boldsymbol{P}]^T$.

Note that the results of Sec. II are applied by mapping the quadrature operators $\boldsymbol{Q}$ and $\boldsymbol{P}$ to the mode operators $\boldsymbol{A}$ and $\boldsymbol{A}^{\dagger}$ via the linear transformation described in Sec. II. In this way, the real Gaussian distribution (12) can be rewritten as a complex Gaussian distribution [64–67] by introducing, for the complex numbers $\alpha = 2^{-1/2}(q + \imath p)$ and $\mu = 2^{-1/2}(\bar{q} + \imath\bar{p})$, the augmented vectors $\check{\boldsymbol{\alpha}} = \boldsymbol{J}\boldsymbol{x}$ and $\check{\boldsymbol{\mu}} = \boldsymbol{J}\bar{\boldsymbol{x}}$, and the augmented covariance matrix $\check{\boldsymbol{C}}_0 = \boldsymbol{J}\boldsymbol{V}\boldsymbol{J}^{\dagger}$, where $\boldsymbol{J}$ is

$$\boldsymbol{J} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & \imath\\ 1 & -\imath\end{bmatrix}.$$

Therefore, the s-ordered characteristic function in the complex variable ξ of a Gaussian state with augmented mean $\check{\boldsymbol{\mu}}$ and augmented covariance matrix $\check{\boldsymbol{C}}_s$ is given by

$$\chi_{\mathrm{G}}(\xi, s) = \exp\left\{-\tfrac{1}{2}\check{\boldsymbol{\xi}}^{\dagger}\boldsymbol{Z}\check{\boldsymbol{C}}_s\boldsymbol{Z}^{\dagger}\check{\boldsymbol{\xi}} + (\boldsymbol{Z}\check{\boldsymbol{\mu}})^{\dagger}\check{\boldsymbol{\xi}}\right\}, \tag{13}$$

where

$$\check{\boldsymbol{C}}_s = \check{\boldsymbol{C}}_0 - \frac{s}{2}\boldsymbol{I} \tag{14}$$

and $\boldsymbol{Z}$ is the Pauli matrix defined as

$$\boldsymbol{Z} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{15}$$

The matrix $\check{\boldsymbol{C}}_0$ represents the augmented covariance matrix of the symmetrically ordered characteristic function. Recall that every Gaussian state can be expressed as a displaced noisy squeezed state with noise parameter $\bar{n}$ and squeezing factor r [29], i.e.,

$$\Xi = \boldsymbol{D}_{\mu}\boldsymbol{R}_{\phi}\boldsymbol{S}_r\,\Xi_{\mathrm{th}}\boldsymbol{S}_r^{\dagger}\boldsymbol{R}_{\phi}^{\dagger}\boldsymbol{D}_{\mu}^{\dagger},$$

where

$$\Xi_{\mathrm{th}} = \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}|n\rangle\langle n|$$

is a thermal state with mean number of photons $\bar{n}$ given by $\mathrm{tr}\{\Xi_{\mathrm{th}}\boldsymbol{A}^{\dagger}\boldsymbol{A}\}$. The matrix $\check{\boldsymbol{C}}_0$ can be rewritten as

$$\check{\boldsymbol{C}}_0 = \left(\bar{n}+\frac{1}{2}\right)\begin{bmatrix}\cosh(2r) & \sinh(2r)e^{-\imath 2\phi}\\ \sinh(2r)e^{\imath 2\phi} & \cosh(2r)\end{bmatrix}. \tag{16}$$

The s-ordered quasiprobability distribution is thus given by the complex Fourier transform of (13), yielding

$$W_{\mathrm{G}}(\alpha, s) = \frac{1}{\pi\sqrt{\det\check{\boldsymbol{C}}_s}}\exp\left\{-\frac{1}{2}(\check{\boldsymbol{\alpha}}-\check{\boldsymbol{\mu}})^{\dagger}\check{\boldsymbol{C}}_s^{-1}(\check{\boldsymbol{\alpha}}-\check{\boldsymbol{\mu}})\right\}, \tag{17}$$

in which, by applying (16) in (14),

$$\det\check{\boldsymbol{C}}_s = \left(\bar{n}+\frac{1-s}{2}\right)^2 - s(2\bar{n}+1)\sinh^2(r). \tag{18}$$

Note that (18) implies that there exists a threshold s_{th} such that $\det\check{\boldsymbol{C}}_s > 0$ only for $s < s_{\mathrm{th}}$ [68,69].[1] By assuming $\det\check{\boldsymbol{C}}_s \neq 0$,

$$\check{\boldsymbol{C}}_s^{-1} = \frac{1}{\det\check{\boldsymbol{C}}_s}\left(\boldsymbol{Z}\check{\boldsymbol{C}}_0\boldsymbol{Z}^{\dagger} - \frac{s}{2}\boldsymbol{I}\right).$$

2. Generalized Hermite polynomials

For a symmetric matrix $\boldsymbol{M}$, the two-variable generalized Hermite polynomials are defined by the generating function [70–72]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{u_1^n u_2^m}{n!m!}H^{\mathrm{g}}_{n,m}(\boldsymbol{x};\boldsymbol{M}) = \exp\{\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{u}+\boldsymbol{x}^{\mathrm{T}}\boldsymbol{u}\}. \tag{19}$$

Note that this paper has implicitly introduced the compact notation $H^{\mathrm{g}}_{n,m}(\boldsymbol{x};\boldsymbol{M})$ to denote $H^{\mathrm{g}}_{n,m}(x_1, M_{11}; x_2, M_{22}|2M_{12})$ in Ref. [71].

Two-variable generalized Hermite polynomials obey the following property that is a generalization of [72, Eq. (7.3.9)].

Lemma 1. For every $\boldsymbol{M}=\boldsymbol{M}^{\mathrm{T}}$ and $\boldsymbol{d}\in\mathbb{C}^2$,

$$\frac{\partial^{m+n}}{\partial x_1^m\partial x_2^n}e^{-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x}+\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}} = (-1)^{m+n}H^{\mathrm{g}}_{m,n}\left(\boldsymbol{M}\boldsymbol{x}-\boldsymbol{d};-\frac{1}{2}\boldsymbol{M}\right)e^{-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x}+\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}}. \tag{20}$$

Proof. From the definition (19) of the two-variable generalized Hermite polynomials, it follows that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{u_1^n u_2^m}{n!m!}e^{-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x}+\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}}H^{\mathrm{g}}_{n,m}\left(\boldsymbol{M}\boldsymbol{x}-\boldsymbol{d};-\frac{1}{2}\boldsymbol{M}\right) = e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{u})^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{x}-\boldsymbol{u})+\boldsymbol{d}^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{u})}. \tag{21}$$

Equation (20) follows from comparing each term in the Taylor expansion of the right side of (21). ⊠

For an augmented Hermitian matrix $\check{\boldsymbol{C}}$, we define new polynomials $\mathscr{H}_{m,n}(\boldsymbol{x};\check{\boldsymbol{C}})$ as

$$\mathscr{H}_{m,n}(\boldsymbol{x};\check{\boldsymbol{C}}) = H^{\mathrm{g}}_{m,n}(\boldsymbol{X}\boldsymbol{x};\boldsymbol{X}\check{\boldsymbol{C}}), \tag{22}$$

where $\boldsymbol{X}$ is the Pauli matrix defined as

$$\boldsymbol{X} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

These polynomials are related to Laguerre polynomials via

$$\mathscr{H}_{m,n}\left(t\boldsymbol{x};-\frac{t}{2}\boldsymbol{I}\right) = n!x_1^{m-n}(-t)^m L_n^{(m-n)}(tx_1x_2). \tag{23}$$

Notice that the two-variable generalized Hermite polynomials are a generalization of the two-variable Hermite polynomials [73–76].

B. Characterization

Consider the initial state Ξ to be Gaussian, as described in Sec. III A. The characterization of the corresponding PVGS $\Xi^{(k)}_{\vec{t}|}$ is given by the following theorem.

Theorem 4. The s-ordered characteristic function $\chi^{(k)}_{\vec{t}|}(\xi, s)$ and quasiprobability distribution $W^{(k)}_{\vec{t}|}(\alpha, s)$ of a PVGS are, respectively, given by

$$\chi^{(k)}_{\vec{t}|}(\xi, s) = \frac{1}{N^{(k)}_{\vec{t}|}}A^{(k)}_{\vec{t}|}(\xi)\,\chi_{\mathrm{G}}(\xi, s) \tag{24a}$$

$$W^{(k)}_{\vec{t}|}(\alpha, s) = \frac{1}{N^{(k)}_{\vec{t}|}}B^{(k)}_{\vec{t}|,s}(\alpha)\,W_{\mathrm{G}}(\alpha, s), \tag{24b}$$

where $\chi_{\mathrm{G}}(\xi, s)$ and $W_{\mathrm{G}}(\alpha, s)$ are the s-ordered characteristic function and quasiprobability distribution of the initial Gaussian state, respectively. The quantity $N^{(k)}_{\vec{t}|}$ and the non-Gaussian functions $A^{(k)}_{\vec{t}|}(\xi)$ and $B^{(k)}_{\vec{t}|,s}(\alpha)$ are given by

$$N^{(k)}_{\vec{t}|} = (-1)^k\mathscr{H}_{k,k}\left(\boldsymbol{Z}\check{\boldsymbol{\mu}};-\frac{1}{2}\boldsymbol{Z}\check{\boldsymbol{C}}_{-t}\boldsymbol{Z}^{\dagger}\right) \tag{25a}$$

$$A^{(k)}_{\vec{t}|}(\xi) = (-1)^k\mathscr{H}_{k,k}\left(\check{\boldsymbol{A}}_t\check{\boldsymbol{\xi}}+\boldsymbol{Z}\check{\boldsymbol{\mu}};-\frac{1}{2}\check{\boldsymbol{A}}_t\right) \tag{25b}$$

$$B^{(k)}_{\vec{t}|,s}(\alpha) = \begin{cases}\left(\frac{s+t}{2}\right)^{2k}\mathscr{H}_{k,k}\left(\check{\boldsymbol{B}}_{t,s}\check{\boldsymbol{\alpha}}-\check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}};-\frac{1}{2}\check{\boldsymbol{B}}_{t,s}\right) & \text{for } s\neq -t\\ |\alpha|^{2k} & \text{for } s=-t,\end{cases} \tag{25c}$$

with

$$\check{\boldsymbol{A}}_t = \boldsymbol{Z}\check{\boldsymbol{C}}_{-t}\boldsymbol{Z}^{\dagger} \tag{26a}$$

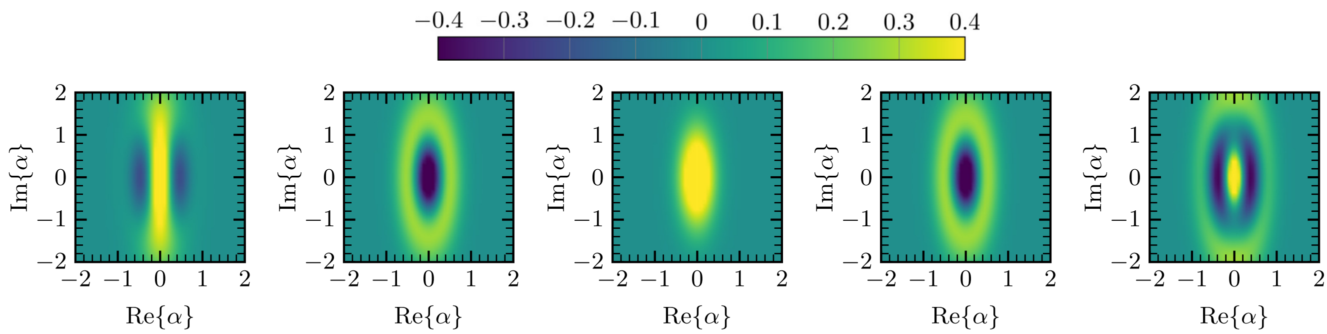
$$\check{\boldsymbol{B}}_{t,s} = \check{\boldsymbol{C}}_s^{-1} + \frac{2}{s+t}\boldsymbol{I}. \tag{26b}$$

Proof. See Appendix D. ⊠

Remark. Theorem 4 reveals the phase-space structure of a PVGS: the s-ordered characteristic function and quasiprobability distribution have a simple canonical product structure. Note that the argument of the multiplicative terms $A^{(k)}_{\vec{t}|}(\xi)$ and $B^{(k)}_{\vec{t}|,s}(\alpha)$ is a linear transformation [66]. In particular, for the s-ordered quasiprobability distribution, a displacement $\check{\boldsymbol{\mu}}$ of the initial Gaussian state produces a corresponding displacement of the multiplicative term, whereas a variation of the covariance matrix $\check{\boldsymbol{C}}_s$ produces a corresponding variation of the augmented matrix $\check{\boldsymbol{B}}_{t,s}$.

Figure 2 shows the Wigner W function $W(\alpha) = W(\alpha, 0)$ of a PVGS for different values of t, k, and $\bar{n}$. Notice that the Wigner function of a PAGS ($t=+1$) gets stretched and loses

[1] In the following, the existence of $W_{\mathrm{G}}(\alpha, s)$ and thus the invertibility of $\check{\boldsymbol{C}}_s$ is assumed.



(a) $k=2,\bar{n}=0,t=-1.$ (b) $k=1,\bar{n}=0,t=-1.$ (c) $k=0,\bar{n}=0.$ (d) $k=1,\bar{n}=0,t=1.$ (e) $k=2,\bar{n}=0,t=1.$

(f) $k=2,\bar{n}=0.2,t=-1.$ (g) $k=1,\bar{n}=0.2,t=-1.$ (h) $k=0,\bar{n}=0.2.$ (i) $k=1,\bar{n}=0.2,t=1.$ (j) $k=2,\bar{n}=0.2,t=1.$

FIG. 2. Wigner W function $W(\alpha)$ of $\Xi_{\vec{t}\rceil}^{(k)}$ for different values of k and $\bar{n}$ with $\mu=0$, $\phi=0$, and $r=-0.5$.

its negativity as $\bar{n}$ increases. Instead, the Wigner function of a PSGS ($t=-1$) has a rather different behavior: as $\bar{n}$ increases, the Wigner function gets stretched, changes its shape, and loses its negativity.

Figure 3 shows the Wigner W function $W(\alpha)=W(\alpha,0)$ of a PVGS for different values of t, k, and $\bar{n}$. In comparison to Fig. 2, it can be observed that the shapes of the function change slightly. This can be attributed to the different shifts of the multiplicative terms in (24b).

Figure 4 shows the Mandel Q parameter of a PVGS, as a function of μ and r, for different values of t, k, and $\bar{n}$. Note that M_{Q} increases as the magnitude of the squeezing parameter r increases and as $\bar{n}$ increases. Note also that the M_{Q} of a PSGS is more affected by noise with respect to a PAGS. Moreover, the range of values of μ and r for which M_{Q} is negative is wider in the case of PAGS compared to PSGS.

C. Special cases

The results of Theorem 4 can be specialized in the presence of a single photon-variation operation ($k=1$) or in the absence of squeezing ($r=0$) as in the following.

1. Single photon-varied Gaussian states

Consider a single PVGS, i.e., $k=1$. This is an important special case since these states are easy to prepare and have been generated in a laboratory [44–48]. Particularizing Theorem 4 to the case $k=1$ leads to the following.

Corollary 1. The s-ordered characteristic function $\chi_{\vec{t}\rceil}^{(1)}(\xi,s)$ and quasiprobability distribution $W_{\vec{t}\rceil}^{(1)}(\alpha,s)$ of a single PVGS are, respectively, found to be

$$\chi_{\vec{t}\rceil}^{(1)}(\xi,s)=\frac{1}{N_{\vec{t}\rceil}^{(1)}}A_{\vec{t}\rceil}^{(1)}(\xi)\,\chi_{\mathrm{G}}(\xi,s) \tag{27a}$$

$$W_{\vec{t}\rceil}^{(1)}(\alpha,s)=\frac{1}{N_{\vec{t}\rceil}^{(1)}}B_{\vec{t}\rceil,s}^{(1)}(\alpha)\,W_{\mathrm{G}}(\alpha,s)\,, \tag{27b}$$

where $\chi_{\mathrm{G}}(\xi,s)$ and $W_{\mathrm{G}}(\alpha,s)$ are the s-ordered characteristic function and quasiprobability distribution of the initial Gaussian state, respectively. The quantity $N_{\vec{t}\rceil}^{(1)}$ and the non-Gaussian functions $A_{\vec{t}\rceil}^{(1)}(\xi)$, and $B_{\vec{t}\rceil,s}^{(1)}(\alpha)$ are given by

$$N_{\vec{t}\rceil}^{(1)}=|\mu|^2+\left(\bar{n}+\frac{1}{2}\right)\cosh(2r)+\frac{t}{2} \tag{28}$$

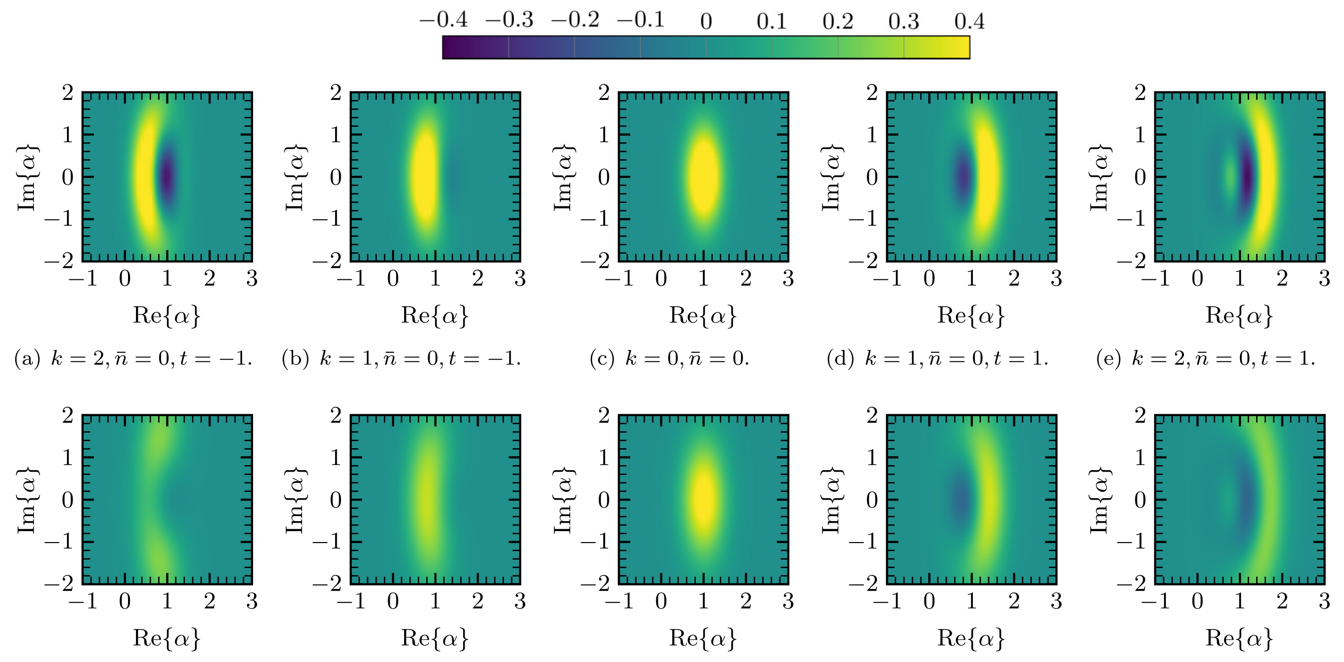
$$A_{\vec{t}\rceil}^{(1)}(\xi)=\frac{1}{2}(\check{\boldsymbol{A}}_t\check{\boldsymbol{\xi}}-\boldsymbol{Z}\check{\boldsymbol{\mu}})^{\dagger}(\check{\boldsymbol{A}}_t\check{\boldsymbol{\xi}}+\boldsymbol{Z}\check{\boldsymbol{\mu}})+[\check{\boldsymbol{A}}_t]_{1,1} \tag{29}$$

$$B_{\vec{t}\rceil,s}^{(1)}(\alpha)=\begin{cases}\frac{1}{2}\left\|\check{\boldsymbol{B}}_{t,s}\check{\boldsymbol{\alpha}}-\check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}}\right\|_2^2-[\check{\boldsymbol{B}}_{t,s}]_{1,1} & \text{for } s\neq -t\\ |\alpha|^2 & \text{for } s=-t,\end{cases} \tag{30}$$

with $\check{\boldsymbol{A}}_t$ and $\check{\boldsymbol{B}}_{t,s}$ given in (26a) and (26b), respectively.

Corollary 1 enables the derivation of a necessary and sufficient condition for the negativity of the quasiprobability distribution for a single PVGS. The negativity of the quasiprobability distribution, in particular, that of the Wigner function ($s=0$) [77], is an important indicator of nonclassicality for any state and of non-Gaussianity for pure states [78,79]. Moreover, negativity of the Wigner function serves as a resource for quantum systems [80] and can provide an advantage in quantum computing [34].

Proposition 1. Let the initial state Ξ be Gaussian, and let $\Xi_{\vec{t}\rceil}^{(1)}$ be the corresponding single PVGS. Then, $W_{\vec{t}\rceil}^{(1)}(\alpha,s)<0$



(a) $k=2, \bar{n}=0, t=-1$. (b) $k=1, \bar{n}=0, t=-1$. (c) $k=0, \bar{n}=0$. (d) $k=1, \bar{n}=0, t=1$. (e) $k=2, \bar{n}=0, t=1$.

(f) $k=2, \bar{n}=0.2, t=-1$. (g) $k=1, \bar{n}=0.2, t=-1$. (h) $k=0, \bar{n}=0.2$. (i) $k=1, \bar{n}=0.2, t=1$. (j) $k=2, \bar{n}=0.2, t=1$.

FIG. 3. Wigner W function $W(\alpha)$ of $\Xi_{\vec{l}}^{(k)}$ for different values of k and $\bar{n}$ with $\mu=1$, $\phi=0$, and $r=-0.5$.

for some $\alpha \in \mathbb{C}$, if and only if

$$[\check{\boldsymbol{B}}_{t,s}]_{1,1} > \tfrac{1}{2}\left\|\check{\boldsymbol{B}}_{t,s}\check{\boldsymbol{B}}_{t,s}^{+}\check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}} - \check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}}\right\|_2^2, \tag{31}$$

where $\check{\boldsymbol{B}}_{t,s}$ is defined in (26b).

Proof. Recall that, from the properties of the Moore-Penrose pseudoinverse, $\check{\boldsymbol{B}}_{t,s}^{+}\check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}}$ is the minimal least-square solution to the linear system $\check{\boldsymbol{B}}_{t,s}\check{\boldsymbol{\alpha}} = \check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}}$ [60,81], i.e., for every $\boldsymbol{z} \in \mathbb{C}^2$, the following bound holds:

$$\left\|\check{\boldsymbol{B}}_{t,s}\boldsymbol{z} - \check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}}\right\|_2 \geqslant \left\|\check{\boldsymbol{B}}_{t,s}\check{\boldsymbol{B}}_{t,s}^{+}\check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}} - \check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}}\right\|_2. \tag{32}$$

From (27b) it follows that $W_{\vec{l}}^{(1)}(\alpha, s) < 0$ if and only if $B_{\vec{l},s}^{(1)}(\alpha) < 0$. From (30), $B_{\vec{l},s}^{(1)}(\alpha) < 0$ if and only if

$$[\check{\boldsymbol{B}}_{t,s}]_{1,1} > \tfrac{1}{2}\left\|\check{\boldsymbol{B}}_{t,s}\check{\boldsymbol{\alpha}} - \check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}}\right\|_2^2. \tag{33}$$

Equation (31) follows by applying (32) in (33). ⊠

Corollary 2. If $\check{\boldsymbol{B}}_{t,s}$ is invertible, a necessary and sufficient condition for the negativity of the quasiprobability distribution $W_{\vec{l}}^{(1)}(\alpha, s)$ is

$$[\check{\boldsymbol{B}}_{t,s}]_{1,1} > 0. \tag{34}$$

Proof. The necessary condition is obtained by noticing that the right-hand side of (31) is non-negative. If $\check{\boldsymbol{B}}_{t,s}$ is invertible, then $\check{\boldsymbol{B}}_{t,s}^{+} = \check{\boldsymbol{B}}_{t,s}^{-1}$ and thus the right-hand side of (31) is equal to zero. ⊠

Remark. This corollary gives a condition for the negativity of the quasiprobability distributions. In particular, by applying (16) in (34) with $s=0$, the condition for the negativity of the Wigner function can be reduced to

$$\frac{\cosh(2r)}{2\bar{n}+1} + t > 0. \tag{35}$$

Note that this condition is always satisfied by PAGSs ($t=+1$). Conversely, for PSGSs ($t=-1$), the condition is satisfied only if $\cosh(2r) > 2\bar{n}+1$. This means that, for PSGSs, thermal noise has to be compensated by squeezing to guarantee the negativity of the Wigner function. This condition generalizes the condition for the case of no displacement, i.e., $\mu=0$, provided in [43]. Therefore, (35) can be used to design PSGSs with a negative Wigner function.

2. Photon-varied coherent states

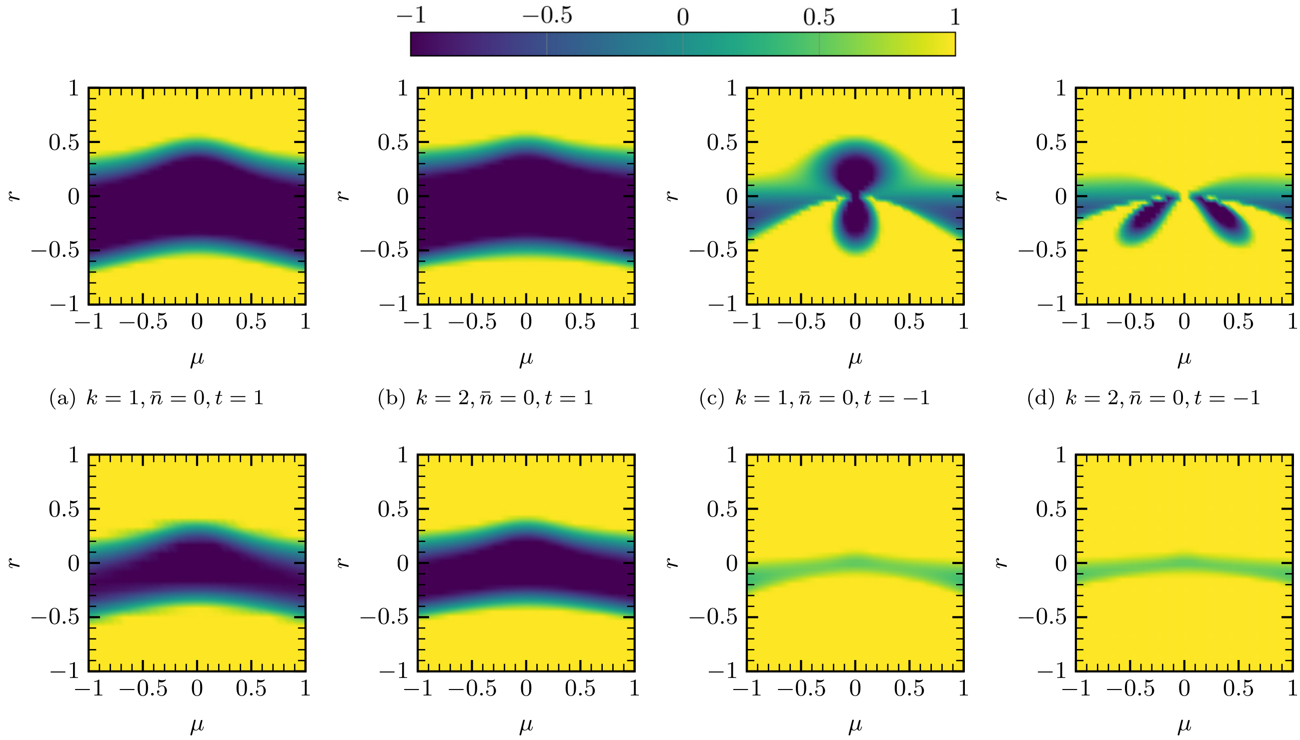
Consider a PVCS, i.e., the initial state Ξ is a coherent state ($r=0$ in (16)). This is another important special case since coherent states can be easily prepared. For a PVCS the representation of the state $\Xi_{\vec{l}}^{(k)}$ reduces to the following simple structure.

Corollary 3. The s-ordered characteristic function $\chi_{\vec{l}}^{(k)}(\xi, s)$ and quasiprobability distribution $W_{\vec{l}}^{(k)}(\alpha, s)$ of a PVCS are, respectively, found to be

$$\chi_{\vec{l}}^{(k)}(\xi, s) = \frac{1}{N_{\vec{l}}^{(k)}} A_{\vec{l}}^{(k)}(\xi)\, \chi_{\mathrm{G}}(\xi, s) \tag{36a}$$

$$W_{\vec{l}}^{(k)}(\alpha, s) = \frac{1}{N_{\vec{l}}^{(k)}} B_{\vec{l},s}^{(k)}(\alpha)\, W_{\mathrm{G}}(\alpha, s), \tag{36b}$$

where $\chi_{\mathrm{G}}(\xi, s)$ and $W_{\mathrm{G}}(\alpha, s)$ are the s-ordered characteristic function and quasiprobability distribution of the initial Gaussian state, respectively. The quantity $N_{\vec{l}}^{(k)}$ and the non-Gaussian functions $A_{\vec{l}}^{(k)}(\xi)$, and $B_{\vec{l},s}^{(k)}(\alpha)$ are



(a) $k = 1, \bar{n} = 0, t = 1$ (b) $k = 2, \bar{n} = 0, t = 1$ (c) $k = 1, \bar{n} = 0, t = -1$ (d) $k = 2, \bar{n} = 0, t = -1$

(e) $k = 1, \bar{n} = 0.2, t = 1$ (f) $k = 2, \bar{n} = 0.2, t = 1$ (g) $k = 1, \bar{n} = 0.2, t = -1$ (h) $k = 2, \bar{n} = 0.2, t = -1$

FIG. 4. Mandel Q parameter of $\boldsymbol{\Xi}_{\overline{t}|}^{(k)}$, as a function of μ and r, for different values of k and $\bar{n}$, with $\phi = 0$.

given by

$$N_{\overline{t}|}^{(k)} = k!\left(\bar{n} + \frac{1+t}{2}\right)^k L_k\left(-\frac{|\mu|^2}{\bar{n} + \frac{1+t}{2}}\right) \tag{37}$$

$$A_{\overline{t}|}^{(k)}(\xi) = k!\left(\bar{n} + \frac{1+t}{2}\right)^k L_k\left(\left(\bar{n} + \frac{1+t}{2}\right)\left(\xi + \frac{\mu}{\bar{n} + \frac{1+t}{2}}\right)\left(\xi^* - \frac{\mu^*}{\bar{n} + \frac{1+t}{2}}\right)\right) \tag{38}$$

and $B_{\overline{t}|,-t}(\alpha) = |\alpha|^{2k}$ for $s = -t$, while for $s \neq -t$

$$B_{\overline{t}|,s}^{(k)}(\alpha) = (-1)^k k!\left(\frac{\left(\bar{n} + \frac{1+t}{2}\right)(s+t)}{2\bar{n} + 1 - s}\right)^k L_k\left(\frac{4\left(\bar{n} + \frac{1+t}{2}\right)}{(s+t)(2\bar{n} + 1 - s)}\left|\alpha - \frac{s+t}{2\bar{n} + 1 + t}\mu\right|^2\right). \tag{39}$$

Proof. See Appendix E. ⊠

IV. CONCLUSION

This paper introduced the class of PVQSs, generated by photon-addition and photon-subtraction operations on any initial quantum state, and developed a framework for their unified characterization in terms of characteristic function, quasiprobability distribution, and Fock representation. In the case of PVGSs, where the initial state is Gaussian, the characterization is found to be in a simple canonical product structure for both the characteristic function and the quasiprobability distribution. Necessary and sufficient conditions are also given for the negativity of the quasiprobability distributions. The findings of this paper open the way to the use of PVQSs with desirable nonclassical properties in various applications of quantum systems and networks.

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APPENDIX A: PROOF OF THEOREM 1

By using the anti-normally ordered form for the displacement operator, the s-ordered characteristic function associated with the state $\boldsymbol{\Xi}_{+}^{(k)}$ can be written as

$$\chi_{+}^{(k)}(\xi, s) = \frac{1}{N_{+}^{(k)}} e^{\frac{s+1}{2}|\xi|^2} \operatorname{tr}\left\{\boldsymbol{\Xi} \boldsymbol{A}^k e^{-\xi^* \boldsymbol{A}} e^{\xi \boldsymbol{A}^\dagger} (\boldsymbol{A}^\dagger)^k\right\}. \tag{A1}$$

By applying the identity

$$\frac{\partial^{2k}}{\partial\xi^k\partial\xi^{*k}}\mathrm{tr}\{\Xi e^{-\xi^* A}e^{\xi A^\dagger}\} = (-1)^k\mathrm{tr}\{\Xi A^k e^{-\xi^* A}e^{\xi A^\dagger}(A^\dagger)^k\},$$

(A1) can be rewritten in terms of Wirtinger derivatives as

$$\chi_+^{(k)}(\xi,s) = \frac{(-1)^k}{N_+^{(k)}}e^{\frac{s+1}{2}|\xi|^2}\frac{\partial^{2k}}{\partial\xi^k\partial\xi^{*k}}\mathrm{tr}\{\Xi e^{-\xi^* A}e^{\xi A^\dagger}\}. \quad (A2)$$

Equation (4), for $t=+1$, is obtained from (A2) by expressing the displacement operator in the symmetrically ordered form.

By using the normally ordered form for the displacement operator, the s-ordered characteristic function associated with the state $\Xi_-^{(k)}$ can be written as

$$\chi_-^{(k)}(\xi,s) = \frac{1}{N_-^{(k)}}e^{\frac{s-1}{2}|\xi|^2}\mathrm{tr}\{\Xi(A^\dagger)^k e^{\xi A^\dagger}e^{-\xi^* A}A^k\}. \quad (A3)$$

By applying the identity

$$\frac{\partial^{2k}}{\partial\xi^k\partial\xi^{*k}}\mathrm{tr}\{\Xi e^{\xi A^\dagger}e^{-\xi^* A}\} = (-1)^k\mathrm{tr}\{\Xi(A^\dagger)^k e^{\xi A^\dagger}e^{-\xi^* A}A^k\},$$

(A3) can be rewritten in terms of Wirtinger derivatives as

$$\chi_-^{(k)}(\xi,s) = \frac{(-1)^k}{N_-^{(k)}}e^{\frac{s-1}{2}|\xi|^2}\frac{\partial^{2k}}{\partial\xi^k\partial\xi^{*k}}\mathrm{tr}\{\Xi e^{\xi A^\dagger}e^{-\xi^* A}\}. \quad (A4)$$

Equation (4), for $t=-1$, is obtained from (A4) by expressing the displacement operator in the symmetrically ordered form.

APPENDIX B: PROOF OF THEOREM 2

The s-ordered quasiprobability distribution $W(\alpha,s)$ associated with the state $\Xi_{\overline{t}|}^{(k)}$ is given by the Fourier transform of (4), i.e.,

$$W_{\overline{t}|}^{(k)}(\alpha,s) = \frac{(-1)^k}{\pi^2 N_{\overline{t}|}^{(k)}}\int_{\mathbb{R}^2} e^{\frac{s+t}{2}|\xi|^2+\alpha\xi^*-\alpha^*\xi} \times \frac{\partial^{2k}}{\partial\xi^k\partial\xi^{*k}}\chi(\xi,s)\,e^{-\frac{s+t}{2}|\xi|^2}d^2\xi. \quad (B1)$$

Integration by parts in (B1) then leads to

$$W_{\overline{t}|}^{(k)}(\alpha,s) = \frac{(-1)^k}{\pi^2 N_{\overline{t}|}^{(k)}}\int_{\mathbb{R}^2}\chi(\xi,s)\,e^{-\frac{s+t}{2}|\xi|^2} \times \underbrace{\frac{\partial^{2k}}{\partial\xi^k\partial\xi^{*k}}e^{\frac{s+t}{2}|\xi|^2+\alpha\xi^*-\alpha^*\xi}}_{\kappa_{s,t}(\alpha,\xi;k)}\,d^2\xi. \quad (B2)$$

By assuming $s\neq -t$, the term $\kappa_{s,t}(\alpha,\xi;k)$ can be written as follows:

$$\begin{aligned}
\kappa_{s,t}(\alpha,\xi;k) &= \frac{\partial^{2k}}{\partial\xi^k\partial\xi^{*k}}\exp\left\{\frac{s+t}{2}\left(\xi+\frac{2}{s+t}\alpha\right)\left(\xi^*-\frac{2}{s+t}\alpha^*\right)\right\}\exp\left\{\frac{2|\alpha|^2}{s+t}\right\}\\
&= \left(\frac{s+t}{2}\right)^k k!\,L_k\left(-\frac{s+t}{2}\left(\xi+\frac{2}{s+t}\alpha\right)\left(\xi^*-\frac{2}{s+t}\alpha^*\right)\right)\exp\left\{\frac{s+t}{2}\left(\xi+\frac{2}{s+t}\alpha\right)\left(\xi^*-\frac{2}{s+t}\alpha^*\right)\right\}\exp\left\{\frac{2|\alpha|^2}{s+t}\right\}\\
&= \left(\frac{s+t}{2}\right)^k k!\,L_k\left(\frac{2}{s+t}\left(\alpha+\frac{s+t}{2}\xi\right)\left(\alpha^*-\frac{s+t}{2}\xi^*\right)\right)\exp\left\{\frac{2}{s+t}\left(\alpha+\frac{s+t}{2}\xi\right)\left(\alpha^*-\frac{s+t}{2}\xi^*\right)\right\}\exp\left\{\frac{2|\alpha|^2}{s+t}\right\}\\
&= (-1)^k\left(\frac{s+t}{2}\right)^{2k}\exp\left\{\frac{2|\alpha|^2}{s+t}\right\}\frac{\partial^{2k}}{\partial\alpha^k\partial\alpha^{*k}}\exp\left\{-\frac{2}{s+t}|\alpha|^2+\alpha\xi^*-\alpha^*\xi\right\}\exp\left\{\frac{s+t}{2}|\xi|^2\right\},
\end{aligned} \quad (B3)$$

where the first equality follows from simple algebra, the second equality from the definition of Laguerre polynomials, the third equality from simple algebra, and the last equality by applying the definition of Laguerre polynomials with respect to α. Equation (6) then follows by applying (B3) in (B2) and applying the definition of s-ordered quasiprobability distribution. Equation (7) follows immediately by noting that $\kappa_{-t,t}(\alpha,\xi;k) = (-1)^k|\alpha|^2\exp\{\alpha\xi^*-\alpha^*\xi\}$.

APPENDIX C: PROOF OF THEOREM 3

Recall that, for every Fock state $|n\rangle$ with $n\in\mathbb{N}$,

$$(A^\dagger)^k|n\rangle = \sqrt{\frac{(n+k)!}{n!}}|n+k\rangle \quad \text{for } k\in\mathbb{N} \quad (C1)$$

and

$$(A)^k|n\rangle = \begin{cases}0 & \text{for } k>n\\ \sqrt{\frac{n!}{(n-k)!}}|n-k\rangle & \text{otherwise.}\end{cases} \quad (C2)$$

Applying (2) or (1) for the state $\Xi_{\overline{t}|}^{(k)}$ into the left side of (9), together with (C2) or (C1), gives the desired result.

APPENDIX D: PROOF OF THEOREM 4

Consider now a given Gaussian state Ξ with $\check{\mu}\neq\mathbf{0}$. The s-ordered characteristic function associated with the state $\Xi_{\overline{t}|}^{(k)}$ is obtained by applying (13) in (4), together with (26a) to obtain

$$\chi_{\bar{\eta}}^{(k)}(\xi, s) = \frac{(-1)^k}{N_{\bar{\eta}}^{(k)}} \exp\left\{\frac{1}{2}\check{\boldsymbol{\xi}}^\dagger\left(\frac{s+t}{2}\boldsymbol{I}\right)\check{\boldsymbol{\xi}}\right\} \times \frac{\partial^{2k}}{\partial\alpha^k\partial\alpha^{*k}} \exp\left\{-\frac{1}{2}\check{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{X}\check{\boldsymbol{A}}_t\check{\boldsymbol{\xi}} + \boldsymbol{\sigma}^{\mathrm{T}}\check{\boldsymbol{\xi}}\right\}, \quad \text{(D1)}$$

where $\boldsymbol{\sigma} = \boldsymbol{Z}\boldsymbol{X}\check{\boldsymbol{\mu}} = -\boldsymbol{X}\boldsymbol{Z}\check{\boldsymbol{\mu}}$.

Equation (24a) is obtained by applying (22) and (25b) in (D1). Equation (25a) is obtained by imposing the normalization condition $\chi_{\bar{\eta}}^{(k)}(0, s) = 1$ in (24a), together with (25b).

The s-ordered quasiprobability distribution, for $s \neq -t$, can be derived by applying (17) in (6), together with (26b) to obtain

$$W_{\bar{\eta}}^{(k)}(\alpha, s) = \frac{1}{N_{\bar{\eta}}^{(k)}\pi\sqrt{\det\check{\boldsymbol{C}}_s}}\left(\frac{s+t}{2}\right)^{2k} \times \exp\left\{\frac{1}{2}\left[\check{\boldsymbol{\alpha}}^\dagger\left(\frac{2}{s+t}\boldsymbol{I}\right)\check{\boldsymbol{\alpha}} + \check{\boldsymbol{\mu}}^\dagger\check{\boldsymbol{C}}_s^{-1}\check{\boldsymbol{\mu}}\right]\right\} \times \frac{\partial^{2k}}{\partial\alpha^k\partial\alpha^{*k}} \exp\left\{-\frac{1}{2}\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}\check{\boldsymbol{B}}_{t,s}\boldsymbol{\alpha} + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{X}\check{\boldsymbol{C}}_s^{-1}\boldsymbol{\alpha}\right\}. \quad \text{(D2)}$$

Equation (24b), for $s \neq -t$, is obtained by applying (22) and (20) in (D2), together with (25c). Equation (24b), for $s = -t$, is obtained by applying (17) in (7), together with (25c).

APPENDIX E: PROOF OF CORROLLARY 3

This proof requires the following corollaries of Theorem 4.

Corollary 4. Under the assumption of Theorem 4 and if $\check{\boldsymbol{A}}_t$ is invertible, then (25b) becomes

$$A_{\bar{\eta}}^{(k)}(\xi) = (-1)^k\mathscr{H}_{k,k}\left(\check{\boldsymbol{A}}_t(\check{\boldsymbol{\xi}} + \boldsymbol{\beta}_t); -\tfrac{1}{2}\check{\boldsymbol{A}}_t\right), \quad \text{(E1)}$$

where

$$\boldsymbol{\beta}_t = \boldsymbol{Z}\check{\boldsymbol{C}}_{-t}^{-1}\check{\boldsymbol{\mu}}. \quad \text{(E2)}$$

Corollary 5. Under the assumption of Theorem 4 and if $\check{\boldsymbol{B}}_{t,s}$ is invertible, then (25c) becomes

$$B_{\bar{\eta},s}^{(k)}(z) = \begin{cases} \left(\frac{s+t}{2}\right)^{2k}\mathscr{H}_{k,k}\left(\check{\boldsymbol{B}}_{t,s}(\check{\boldsymbol{\alpha}} - \check{\boldsymbol{\gamma}}_{t,s}); -\frac{1}{2}\check{\boldsymbol{B}}_{t,s}\right) & \\ & \text{for } s \neq -t \\ |\alpha|^{2k} & \text{for } s = -t, \end{cases} \quad \text{(E3)}$$

where

$$\check{\boldsymbol{\gamma}}_{t,s} = \frac{s+t}{2}\check{\boldsymbol{C}}_{-t}^{-1}\check{\boldsymbol{\mu}}. \quad \text{(E4)}$$

Consider now a coherent state, where the augmented covariance matrix $\check{\boldsymbol{C}}_s$ is given by applying (16) with $r = 0$ in (14), which gives

$$\check{\boldsymbol{C}}_s = \left(\bar{n} + \frac{1-s}{2}\right)\boldsymbol{I}, \quad \text{(E5)}$$

for which the matrix $\check{\boldsymbol{A}}_t$ defined in (26a) is found to be

$$\check{\boldsymbol{A}}_t = \left(\bar{n} + \frac{1+t}{2}\right)\boldsymbol{I}. \quad \text{(E6)}$$

From (E5), the matrix $\check{\boldsymbol{C}}_s^{-1}$ is easily found to be

$$\check{\boldsymbol{C}}_s^{-1} = \frac{2}{2\bar{n}+1-s}\boldsymbol{I}. \quad \text{(E7)}$$

Since $\check{\boldsymbol{A}}_t$ is invertible, by applying (23) and (E6) into (E1) we obtain

$$A_{\bar{\eta}}^{(k)} = k!\left(\bar{n} + \frac{1+t}{2}\right)^k \times L_k\left(\left(\bar{n} + \frac{1+t}{2}\right)(z + [\boldsymbol{\beta}_t]_1)(z^* + [\boldsymbol{\beta}_t]_2)\right). \quad \text{(E8)}$$

The vector $\boldsymbol{\beta}_t$ is obtained by applying (E7) with $s = -t$ in (E2) to obtain

$$\boldsymbol{\beta}_t = \left(\bar{n} + \frac{1+t}{2}\right)^{-1}\begin{bmatrix}\mu \\ -\mu^*\end{bmatrix}. \quad \text{(E9)}$$

Equation (38) is obtained by applying (E9) in (E8). By applying (E7) in (26b), the matrix $\check{\boldsymbol{B}}_{t,s}$ is found to be

$$\check{\boldsymbol{B}}_{t,s} = \frac{4\left(\bar{n} + \frac{1+t}{2}\right)}{(s+t)(2\bar{n}+1-s)}\boldsymbol{I}. \quad \text{(E10)}$$

Since $\check{\boldsymbol{B}}_{t,s}$ is invertible, by applying (23) and (E10) in (E3) we obtain

$$B_{\bar{\eta},s}^{(s)} = (-1)^k k!\left[\frac{(s+t)\left(\bar{n} + \frac{1+t}{2}\right)}{2\bar{n}+1-s}\right]^k \times L_k\left(\frac{4\left(\bar{n} + \frac{1+t}{2}\right)}{(s+t)(2\bar{n}+1-s)}|\alpha - \gamma_{t,s}|^2\right), \quad \text{(E11)}$$

where $\gamma_{t,s}$ is the complex number associated with the augmented vector $\check{\boldsymbol{\gamma}}_{t,s}$, obtained by applying (E7) in (E4), i.e.,

$$\check{\boldsymbol{\gamma}}_{t,s} = \frac{s+t}{2\bar{n}+1+t}\boldsymbol{I}. \quad \text{(E12)}$$

Equation (39) follows by applying (E12) in (E11).

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