Message Passing Algorithms for Scalable Multitarget Tracking — Supplementary Material

Florian Meyer, Thomas Kropfreiter, Jason L. Williams, Roslyn A. Lau,
Franz Hlawatsch, Paolo Braca, and Moe Z. Win

July 2, 2019

This manuscript provides an errata list and three derivations for the publication, “Message Passing Algorithms for Scalable Multitarget Tracking” by the same authors [1].

1 Errata

Page/Column of [1]	Original Text	Corrected Text
223/1	A6) The number of clutter measurements at sensor s and time k is Poisson distributed with mean $\mu_{\mathrm{c}}^{(s)}$. It is furthermore independent across the sensors s and independent of the number of targets that are detected at sensor s.	A6) The number of clutter measurements at sensor s and time k is Poisson distributed with mean $\mu_{\mathrm{c}}^{(s)}$. It is independent—both unconditionally and conditionally on the target states—across the sensors s. Conditioned on the current target states, it is furthermore independent of the indices of the targets that are detected at sensor s. Finally, it is also statistically independent of the current target states.
228/1	However, $\mathbf{a}_{k,s}$ is unknown and thus considered as a latent random variable in the inference problem.	However, $\mathbf{a}_{k,s}$ is unknown and thus considered as a latent random variable in the inference problem. The DA vector $\mathbf{a}_{k,s}$ is introduced such that it inherits the statistical properties of the corresponding measurement vector $\mathbf{z}_{k,s}$; in particular, any variable that is statistically independent of $\mathbf{z}_{k,s}$ is also statistically independent of $\mathbf{a}_{k,s}$.
232/2	This iterative algorithm is initialized by $\varphi_k^{[0](i\to m)} = \beta_k^{(i)}(m)/\beta_k^{(i)}(0)$.	This iterative algorithm is initialized by $\varphi_k^{[0](i\to m)} = \beta_k^{(i)}(m)/\sum_{\substack{m'=0\\m'\neq m}}^{M_k} \beta_k^{(i)}(m')$.
237/2	Vu4) The number of newly detected targets at time k and sensor s is *a priori* (i.e., before the measurements are observed) Poisson distributed with mean $\mu_{\mathrm{n}}^{(s)}$. It is furthermore independent of the number of clutter measurements and of the number of survived targets.	Vu4) The number of newly detected targets at time k and sensor s is *a priori* (i.e., before the measurements are observed) Poisson distributed with mean $\mu_{\mathrm{n}}^{(s)}$. Conditioned on the states of the survived targets, it is furthermore independent of the number of clutter measurements and of the indices of the targets that are detected at sensor s. Finally, it is also statistically independent of the states of the survived targets.
246/1	[8]If the label is viewed as a proxy for the trajectory, the formulation in [153] provides a similar result directly via a Bayes update.	[8]If the label is viewed as a proxy for the trajectory, the formulation in [152] provides a similar result directly via a Bayes update.

2 Derivation of [1, Eq. (11)]

In this section, we derive the expression of $p(\boldsymbol{a}_{k,s}, m_{k,s}|\boldsymbol{x}_k)$ in [1, Eq. (11)], for which a less detailed derivation was previously given in [2]. Here, $\mathbf{x}_k = \big[\mathbf{x}_k^{(1)\mathrm{T}} \cdots \mathbf{x}_k^{(n_\mathrm{t})\mathrm{T}}\big]^\mathrm{T}$ is the joint state vector, representing the states of the n_t targets at time k; $\mathsf{m}_{k,s}$ is the number of measurements; and $\mathbf{a}_{k,s} = \big[\mathsf{a}_{k,s}^{(1)} \cdots \mathsf{a}_{k,s}^{(n_\mathrm{t})}\big]^\mathrm{T}$ is the target-measurement association vector, whose entries $\mathsf{a}_{k,s}^{(i)}$ are given by [1, Eq. (10)]. For the sake of notational simplicity, we omit the sensor index s in the following.

We start by defining the n_t-dimensional binary vector $\boldsymbol{\theta}_k = \big[\theta_k^{(1)} \cdots \theta_k^{(n_\mathrm{t})}\big]^\mathrm{T}$ that indicates which targets $i \in \{1, \ldots, n_\mathrm{t}\}$ are detected ($\theta_k^{(i)} = 1$) or missed ($\theta_k^{(i)} = 0$). Using Assumption A5 from [1, Sec. I-C], the pmf of $\boldsymbol{\theta}_k$ given $\mathbf{x}_k$ is obtained as

$$p(\boldsymbol{\theta}_k|\boldsymbol{x}_k) = \prod_{i=1}^{n_\mathrm{t}} p\big(\theta_k^{(i)}\big|\boldsymbol{x}_k^{(i)}\big) \tag{1}$$

with

$$p\big(\theta_k^{(i)}\big|\boldsymbol{x}_k^{(i)}\big) = \begin{cases} p_\mathrm{d}\big(\boldsymbol{x}_k^{(i)}\big), & \theta_k^{(i)} = 1 \\ 1 - p_\mathrm{d}\big(\boldsymbol{x}_k^{(i)}\big), & \theta_k^{(i)} = 0. \end{cases} \tag{2}$$

Furthermore, let $\mathsf{n}_k^\mathrm{d}(\boldsymbol{\theta}_k) \triangleq \sum_{i=1}^{n_\mathrm{t}} \theta_k^{(i)}$ denote the number of detected targets. On the other hand, according to Assumption A6 from [1, Sec. I-C], the number of clutter measurements n_k^c is Poisson distributed, with pmf

$$p(n_k^\mathrm{c}) = \frac{\mu_\mathrm{c}^{n_k^\mathrm{c}}}{n_k^\mathrm{c}!} e^{-\mu_\mathrm{c}}, \tag{3}$$

where μ_c is the mean number of clutter measurements. Hence, the total number of measurements m_k is given by

$$\mathsf{m}_k = \mathsf{n}_k^\mathrm{d}(\boldsymbol{\theta}_k) + \mathsf{n}_k^\mathrm{c} = \sum_{i=1}^{n_\mathrm{t}} \theta_k^{(i)} + \mathsf{n}_k^\mathrm{c}. \tag{4}$$

Note that the information carried by $\theta_k^{(i)}$, i.e., whether target $i \in \{1, \ldots, n_t\}$ is detected or not, is also contained in the association variable $\mathsf{a}_k^{(i)}$. More precisely, $\theta_k^{(i)} = 1$ is equivalent to $\mathsf{a}_k^{(i)} \neq 0$ and $\theta_k^{(i)} = 0$ is equivalent to $\mathsf{a}_k^{(i)} = 0$. Because of this equivalence and using (4), there is a one-to-one relation between $(\mathbf{a}_k, \boldsymbol{\theta}_k, \mathsf{n}_k^\mathrm{c})$ and $(\mathbf{a}_k, \mathsf{m}_k)$, and therefore we have

$$p(\boldsymbol{a}_k, \boldsymbol{\theta}_k, n_k^\mathrm{c}|\boldsymbol{x}_k) = p(\boldsymbol{a}_k, m_k|\boldsymbol{x}_k). \tag{5}$$

For now, we consider only target-measurement associations $\boldsymbol{a}_k$ that satisfy Assumption A4 from [1, Sec. I-C], which postulates that a measurement cannot originate from more than one target simultaneously and a target can generate at most one measurement. We have

$$p(\boldsymbol{a}_k, \boldsymbol{\theta}_k, n_k^\mathrm{c}|\boldsymbol{x}_k) = p(\boldsymbol{a}_k|\boldsymbol{\theta}_k, n_k^\mathrm{c}, \boldsymbol{x}_k) p(\boldsymbol{\theta}_k, n_k^\mathrm{c}|\boldsymbol{x}_k) = p(\boldsymbol{a}_k|\boldsymbol{\theta}_k, n_k^\mathrm{c}, \boldsymbol{x}_k) p(\boldsymbol{\theta}_k|\boldsymbol{x}_k) p(n_k^\mathrm{c}). \tag{6}$$

Here, in the second step, we used Assumption A6 from [1, Sec. I-C] in the corrected form provided in Section 1, i.e., the assumptions that (i) $\boldsymbol{\theta}_k$ and n_k^c are conditionally independent given $\mathbf{x}_k = \boldsymbol{x}_k$, and thus $p(\boldsymbol{\theta}_k, n_k^\mathrm{c}|\boldsymbol{x}_k) = p(\boldsymbol{\theta}_k|\boldsymbol{x}_k)\, p(n_k^\mathrm{c}|\boldsymbol{x}_k)$, and (ii) n_k^c is independent of $\mathbf{x}_k$, and thus $p(n_k^\mathrm{c}|\boldsymbol{x}_k) = p(n_k^\mathrm{c})$. Expressions of $p(\boldsymbol{\theta}_k|\boldsymbol{x}_k)$ and $p(n_k^\mathrm{c})$ are given by (1), (2) and by (3), respectively.

It remains to find an expression of $p(\boldsymbol{a}_k|\boldsymbol{\theta}_k, n_k^\mathrm{c}, \boldsymbol{x}_k)$ in (6). For given $\boldsymbol{\theta}_k$, n_k^c, and $\boldsymbol{x}_k$, the number $N_{\boldsymbol{a}_k}$ of target-measurement associations $\boldsymbol{a}_k$ that satisfy Assumption A4 is equivalent to the number of draws of $n_k^\mathrm{d}(\boldsymbol{\theta}_k)$ measurements out of the m_k existing measurements, where the draws are without replacement and with the drawing order respected. (Note that the drawing order has to be respected, because the same overall draw but with a different drawing order corresponds to a different target-measurement association $\boldsymbol{a}_k$.) By using basic results of combinatorics, we obtain

$$N_{\boldsymbol{a}_k} = \frac{m_k!}{\big(m_k - n_k^{\mathrm{d}}(\boldsymbol{\theta}_k)\big)!} = \frac{m_k!}{n_k^{\mathrm{c}}!}\,, \tag{7}$$

where (4) was used in the last step. We assume that each of the $N_{\boldsymbol{a}_k}$ possible draws is equally likely, which yields

$$p(\boldsymbol{a}_k|\boldsymbol{\theta}_k, n_k^{\mathrm{c}}, \boldsymbol{x}_k) = \frac{1}{N_{\boldsymbol{a}_k}} = \frac{n_k^{\mathrm{c}}!}{m_k!}\,. \tag{8}$$

By inserting (8), (1), and (3) into (6), we obtain

$$p(\boldsymbol{a}_k, \boldsymbol{\theta}_k, n_k^{\mathrm{c}}|\boldsymbol{x}_k) \;=\; \frac{n_k^{\mathrm{c}}!}{m_k!}\left(\prod_{i=1}^{n_{\mathrm{t}}} p\big(\theta_k^{(i)}\big|\boldsymbol{x}_k^{(i)}\big)\right)\frac{\mu_{\mathrm{c}}^{n_k^{\mathrm{c}}}}{n_k^{\mathrm{c}}!}\,e^{-\mu_{\mathrm{c}}} \;=\; \frac{e^{-\mu_{\mathrm{c}}}\,\mu_{\mathrm{c}}^{n_k^{\mathrm{c}}}}{m_k!}\prod_{i=1}^{n_{\mathrm{t}}} p\big(\theta_k^{(i)}\big|\boldsymbol{x}_k^{(i)}\big)\,.$$

Inserting (2) then gives

$$p(\boldsymbol{a}_k, \boldsymbol{\theta}_k, n_k^{\mathrm{c}}|\boldsymbol{x}_k) = \frac{e^{-\mu_{\mathrm{c}}}\,\mu_{\mathrm{c}}^{n_k^{\mathrm{c}}}}{m_k!}\left(\prod_{i:\theta_k^{(i)}=0}\big(1 - p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(i)}\big)\big)\right)\prod_{j:\theta_k^{(j)}=1} p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(j)}\big). \tag{9}$$

Let us introduce the index set $\mathcal{D}_{\boldsymbol{a}_k} \triangleq \big\{i \in \{1,\dots,n_{\mathrm{t}}\} : a_k^{(i)} \neq 0\big\}$ corresponding to the set of detected targets. Here, we recall that $a_k^{(i)} = 0$ indicates that target i is not detected and $a_k^{(i)} \neq 0$ indicates that it is detected. Thus, $i \notin \mathcal{D}_{\boldsymbol{a}_k}$ is equivalent to $\theta_k^{(i)} = 0$ and $i \in \mathcal{D}_{\boldsymbol{a}_k}$ to $\theta_k^{(i)} = 1$. Using this equivalence as well as the identities $|\mathcal{D}_{\boldsymbol{a}_k}| = n_k^{\mathrm{d}}(\boldsymbol{\theta}_k)$ and (cf. (4)) $n_k^{\mathrm{c}} = m_k - |\mathcal{D}_{\boldsymbol{a}_k}|$, Eq. (9) can be rewritten as

$$\begin{aligned}
p(\boldsymbol{a}_k, \boldsymbol{\theta}_k, n_k^{\mathrm{c}}|\boldsymbol{x}_k) &= \frac{e^{-\mu_{\mathrm{c}}}\,\mu_{\mathrm{c}}^{m_k - |\mathcal{D}_{\boldsymbol{a}_k}|}}{m_k!}\left(\prod_{i\notin\mathcal{D}_{\boldsymbol{a}_k}}\big(1 - p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(i)}\big)\big)\right)\prod_{j\in\mathcal{D}_{\boldsymbol{a}_k}} p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(j)}\big) \\
&= \frac{e^{-\mu_{\mathrm{c}}}\,\mu_{\mathrm{c}}^{m_k}}{m_k!\,\mu_{\mathrm{c}}^{|\mathcal{D}_{\boldsymbol{a}_k}|}}\left(\prod_{i=1}^{n_t}\big(1 - p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(i)}\big)\big)\right)\prod_{j\in\mathcal{D}_{\boldsymbol{a}_k}}\frac{p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(j)}\big)}{1 - p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(j)}\big)} \\
&= \frac{e^{-\mu_{\mathrm{c}}}\,\mu_{\mathrm{c}}^{m_k}}{m_k!}\left(\prod_{i=1}^{n_t}\big(1 - p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(i)}\big)\big)\right)\prod_{j\in\mathcal{D}_{\boldsymbol{a}_k}}\frac{p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(j)}\big)}{\mu_{\mathrm{c}}\big(1 - p_{\mathrm{d}}\big(\boldsymbol{x}_k^{(j)}\big)\big)}\,.
\end{aligned} \tag{10}$$

So far, we considered only target-measurement associations $\boldsymbol{a}_k$ that satisfy Assumption A4 from [1, Sec. I-C]. However, expression (10) can be extended to arbitrary $\boldsymbol{a}_k \in \{0,1,\dots,m_k\}^{n_t}$ by multiplying it by the indicator function $\psi(\boldsymbol{a}_k)$ (see [1, Sec. IV-B]), which enforces Assumption A4. Finally, using (5), we obtain [1, Eq. (11)].

3 Derivation of [1, Eqs. (40) and (41)]

In this section, we derive the expression of $\tilde{f}(\boldsymbol{x}_k|\boldsymbol{a}_k, \boldsymbol{z}_{1:k})$ in [1, Eq. (40)] and the expression of $\tilde{f}\big(\boldsymbol{x}_k^{(i)}\big|a_k^{(i)}, \boldsymbol{z}_{1:k}\big)$ in [1, Eq. (41)]. We start by developing $f(\boldsymbol{x}_k|\boldsymbol{a}_k, \boldsymbol{z}_{1:k})$ as

$$\begin{aligned}
f(\boldsymbol{x}_k|\boldsymbol{a}_k, \boldsymbol{z}_{1:k}) &= f(\boldsymbol{x}_k|\boldsymbol{a}_k, \boldsymbol{z}_{1:k}, m_k) \\
&= f(\boldsymbol{x}_k|\boldsymbol{a}_k, \boldsymbol{z}_k, \boldsymbol{z}_{1:k-1}, m_k)
\end{aligned}$$

$$= \frac{f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}_k, \boldsymbol{z}_{1:k-1}) \, f(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1})}{f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{z}_{1:k-1})}$$

$$= \frac{f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}_k) \, f(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1})}{f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{z}_{1:k-1})},$$

where in the third step, we used Bayes' theorem and in the last step, we used Assumption A11 from [1, Sec. I-C], which implies $f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}_k, \boldsymbol{z}_{1:k-1}) = f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}_k)$. Next, we replace the exact predicted posterior pdf $f(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1})$ by the approximate predicted posterior pdf $\tilde{f}(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1})$ (see [1, Eqs. (37) and (38)]), which leads to

$$\tilde{f}(\boldsymbol{x}_k | \boldsymbol{a}_k, \boldsymbol{z}_{1:k}) \triangleq \frac{f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}_k) \, \tilde{f}(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1})}{f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{z}_{1:k-1})}$$

$$= \frac{f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}_k) \, \tilde{f}(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1})}{\int f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}'_k, \boldsymbol{z}_{1:k-1}) \, f(\boldsymbol{x}'_k | \boldsymbol{z}_{1:k-1}) \, \mathrm{d}\boldsymbol{x}'_k}$$

$$\approx \frac{f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}_k) \, \tilde{f}(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1})}{\int f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}'_k) \, \tilde{f}(\boldsymbol{x}'_k | \boldsymbol{z}_{1:k-1}) \, \mathrm{d}\boldsymbol{x}'_k}. \tag{11}$$

Here, in the second step, we applied the total probability theorem, i.e., $f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{z}_{1:k-1}) = \int f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}_k, \boldsymbol{z}_{1:k-1}) \, f(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1}) \, \mathrm{d}\boldsymbol{x}_k$, and in the third step, we replaced $f(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1})$ by $\tilde{f}(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1})$ and used Assumption A11 from [1, Sec. I-C] again. Furthermore, we have from [1, Eq. (13)] (with the sensor index s dropped)

$$f(\boldsymbol{a}_k, \boldsymbol{z}_k, m_k | \boldsymbol{x}_k) = \frac{e^{-\mu_{\mathrm{c}}}}{m_k!} \left(\prod_{m=1}^{m_k} \mu_{\mathrm{c}} f_{\mathrm{c}}\big(\boldsymbol{z}_k^{(m)}\big) \right) \psi(\boldsymbol{a}_k) \prod_{i=1}^{n_{\mathrm{t}}} g\big(\boldsymbol{x}_k^{(i)}, a_k^{(i)}; \boldsymbol{z}_k\big), \tag{12}$$

where $g\big(\boldsymbol{x}_k^{(i)}, a_k^{(i)}; \boldsymbol{z}_k\big)$ is given by [1, Eq. (14)]. Inserting (12) into (11) and using [1, Eq. (37)], i.e., $\tilde{f}(\boldsymbol{x}_k | \boldsymbol{z}_{1:k-1}) = \prod_{i=1}^{n_{\mathrm{t}}} \tilde{f}\big(\boldsymbol{x}_k^{(i)} | \boldsymbol{z}_{1:k-1}\big)$, gives

$$\tilde{f}(\boldsymbol{x}_k | \boldsymbol{a}_k, \boldsymbol{z}_{1:k}) \approx \frac{\prod_{i=1}^{n_{\mathrm{t}}} g\big(\boldsymbol{x}_k^{(i)}, a_k^{(i)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(i)} | \boldsymbol{z}_{1:k-1}\big)}{\int \cdots \int \prod_{i=1}^{n_{\mathrm{t}}} g\big(\boldsymbol{x}_k^{(i)\prime}, a_k^{(i)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(i)\prime} | \boldsymbol{z}_{1:k-1}\big) \, \mathrm{d}\boldsymbol{x}_k^{(i)\prime}},$$

$$= \prod_{i=1}^{n_{\mathrm{t}}} \frac{g\big(\boldsymbol{x}_k^{(i)}, a_k^{(i)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(i)} | \boldsymbol{z}_{1:k-1}\big)}{\int g\big(\boldsymbol{x}_k^{(i)\prime}, a_k^{(i)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(i)\prime} | \boldsymbol{z}_{1:k-1}\big) \, \mathrm{d}\boldsymbol{x}_k^{(i)\prime}}. \tag{13}$$

Next, we calculate the marginal pdf $\tilde{f}\big(\boldsymbol{x}_k^{(i)} | \boldsymbol{a}_k, \boldsymbol{z}_{1:k}\big)$ as

$$\tilde{f}\big(\boldsymbol{x}_k^{(i)} | \boldsymbol{a}_k, \boldsymbol{z}_{1:k}\big) = \int \tilde{f}(\boldsymbol{x}_k | \boldsymbol{a}_k, \boldsymbol{z}_{1:k}) \, \mathrm{d}\boldsymbol{x}_k^{(\sim i)}, \tag{14}$$

where $\int \ldots \, \mathrm{d}\boldsymbol{x}_k^{(\sim i)}$ denotes integration with respect to all vectors $\boldsymbol{x}_k^{(i')}$, $i' \in \{1, \ldots, n_t\}$ except $\boldsymbol{x}_k^{(i)}$. Inserting (13) into (14) yields

$$\tilde{f}\big(\boldsymbol{x}_k^{(i)} | \boldsymbol{a}_k, \boldsymbol{z}_{1:k}\big) = \left(\prod_{\substack{j=1 \\ j \neq i}}^{n_{\mathrm{t}}} \frac{\int g\big(\boldsymbol{x}_k^{(j)}, a_k^{(j)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(j)} | \boldsymbol{z}_{1:k-1}\big) \, \mathrm{d}\boldsymbol{x}_k^{(j)}}{\int g\big(\boldsymbol{x}_k^{(j)}, a_k^{(j)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(j)} | \boldsymbol{z}_{1:k-1}\big) \, \mathrm{d}\boldsymbol{x}_k^{(j)}} \right) \frac{g\big(\boldsymbol{x}_k^{(i)}, a_k^{(i)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(i)} | \boldsymbol{z}_{1:k-1}\big)}{\int g\big(\boldsymbol{x}_k^{(i)\prime}, a_k^{(i)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(i)\prime} | \boldsymbol{z}_{1:k-1}\big) \, \mathrm{d}\boldsymbol{x}_k^{(i)\prime}}$$

$$= \frac{g\big(\boldsymbol{x}_k^{(i)}, a_k^{(i)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(i)} | \boldsymbol{z}_{1:k-1}\big)}{\int g\big(\boldsymbol{x}_k^{(i)\prime}, a_k^{(i)}; \boldsymbol{z}_k\big) \, \tilde{f}\big(\boldsymbol{x}_k^{(i)\prime} | \boldsymbol{z}_{1:k-1}\big) \, \mathrm{d}\boldsymbol{x}_k^{(i)\prime}}. \tag{15}$$

From the fact that the right-hand side of (15) involves only $a_k^{(i)}$, but not $a_k^{(i')}$ for $i' \in \{1,\ldots,n_t\} \setminus \{i\}$, we conclude that $\tilde{f}\big(\boldsymbol{x}_k^{(i)}\big|\boldsymbol{a}_k, \boldsymbol{z}_{1:k}\big) = \tilde{f}\big(\boldsymbol{x}_k^{(i)}\big|a_k^{(i)}, \boldsymbol{z}_{1:k}\big)$. Thus, (15) is seen to be equal to [1, Eq. (41)]. Finally, inserting (15) into (13) yields [1, Eq. (40)].

4 Derivation of [1, Eq. (56)]

In this section, we show that $p(\boldsymbol{a}_{k,s}, \overline{\boldsymbol{r}}_{k,s}, m_{k,s}|\underline{\boldsymbol{y}}_{k,s})$ is given by expression [1, Eq. (56)]. For the sake of notational simplicity, we again omit the sensor index s in the following. We recall that $\mathbf{y}_k = \big[\underline{\mathbf{y}}_k^{\mathrm{T}}\ \overline{\mathbf{y}}_k^{\mathrm{T}}\big]^{\mathrm{T}}$ is the joint augmented state vector defined in Section [1, Sec. VIII-B], which consists of j_{k-1} augmented states $\underline{\mathbf{y}}_k^{(j)}$, $j = 1,\ldots,j_{k-1}$ of legacy potential targets (PTs), i.e., $\underline{\mathbf{y}}_k = \big[\underline{\mathbf{y}}_k^{(1)\mathrm{T}} \cdots \underline{\mathbf{y}}_k^{(j_{k-1})\mathrm{T}}\big]^{\mathrm{T}}$ (see [1, Sec. VIII-C]), and of m_k augmented states $\overline{\mathbf{y}}_k^{(j)}$, $j = 1,\ldots,m_k$ of new PTs, i.e., $\overline{\mathbf{y}}_k = \big[\overline{\mathbf{y}}_k^{(1)\mathrm{T}} \cdots \overline{\mathbf{y}}_k^{(m_k)\mathrm{T}}\big]^{\mathrm{T}}$. Note that $j_k = j_{k-1} + m_k$ (see [1, Eq. (52)]). Each entry of $\underline{\mathbf{y}}_k$ and $\overline{\mathbf{y}}_k$ is a vector of the form $\big[\mathbf{x}_k^{(j)\mathrm{T}}\ \mathrm{r}_k^{(j)}\big]^{\mathrm{T}}$, where $\mathbf{x}_k^{(j)}$ denotes the PT state and $\mathrm{r}_k^{(j)} \in \{0,1\}$ indicates the PT's existence/nonexistence. We finally note that $p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k, m_k|\underline{\boldsymbol{y}}_k)$ is an extension of $p(\boldsymbol{a}_k, m_k|\boldsymbol{x}_k)$ (cf. Section 2) to the case where the number of targets is unknown and time-varying and, thus, the existence of the targets is uncertain, and where the number of new targets is Poisson distributed with mean μ_{n} (cf. Assumptions Vu1 and Vu4 from Section [1, Sec. VIII-A]).

Similarly to Section 2, we introduce the j_{k-1}-dimensional binary vector $\underline{\boldsymbol{\uptheta}}_k = \big[\underline{\uptheta}_k^{(1)} \cdots \underline{\uptheta}_k^{(j_{k-1})}\big]^{\mathrm{T}}$ that indicates which legacy PTs (with states $\underline{\mathbf{y}}_k^{(j)}$) are detected at time k. Using Assumption A5 from [1, Sec. I-C], the pmf of $\underline{\boldsymbol{\uptheta}}_k$ given $\underline{\boldsymbol{y}}_k$ is

$$p(\underline{\boldsymbol{\theta}}_k|\underline{\boldsymbol{y}}_k) = \prod_{j=1}^{j_{k-1}} p\big(\underline{\theta}_k^{(j)}\big|\underline{\boldsymbol{y}}_k^{(j)}\big) \tag{16}$$

with

$$p\big(\underline{\theta}_k^{(j)}\big|\underline{\boldsymbol{y}}_k^{(j)}\big) = \begin{cases} \underline{r}_k^{(j)} p_{\mathrm{d}}\big(\underline{\boldsymbol{x}}_k^{(j)}\big), & \underline{\theta}_k^{(j)} = 1 \\ 1 - \underline{r}_k^{(j)} p_{\mathrm{d}}\big(\underline{\boldsymbol{x}}_k^{(j)}\big), & \underline{\theta}_k^{(j)} = 0. \end{cases} \tag{17}$$

Here, $\underline{r}_k^{(j)} \in \{0,1\}$ for $j \in \{1,\ldots,j_{k-1}\}$. The number of detected legacy PTs is given by $\mathrm{n}_k^{\mathrm{d}}(\underline{\boldsymbol{\uptheta}}_k) \triangleq \sum_{j=1}^{j_{k-1}} \underline{\uptheta}_k^{(j)}$. Furthermore, as mentioned above, the number of measurements originating from newly detected targets, $\mathrm{n}_k^{\mathrm{n}}$, is Poisson distributed according to

$$p(n_k^{\mathrm{n}}) = \frac{\mu_{\mathrm{n}}^{n_k^{\mathrm{n}}}}{n_k^{\mathrm{n}}!} e^{-\mu_{\mathrm{n}}}. \tag{18}$$

The total number of measurements is given by

$$\mathrm{m}_k = \mathrm{n}_k^{\mathrm{d}}(\underline{\boldsymbol{\uptheta}}_k) + \mathrm{n}_k^{\mathrm{n}} + \mathrm{n}_k^{\mathrm{c}}, \tag{19}$$

where $\mathrm{n}_k^{\mathrm{c}}$ denotes the number of clutter measurements. Analogously to Section 2, the information carried by $\underline{\uptheta}_k^{(j)}$, i.e., whether legacy PT $j \in \{1,\ldots,j_{k-1}\}$ is detected or not, is also contained in the association variable $\mathrm{a}_k^{(j)}$: $\underline{\uptheta}_k^{(j)} = 1$ is equivalent to $\mathrm{a}_k^{(j)} \neq 0$ and $\underline{\uptheta}_k^{(j)} = 0$ is equivalent to $\mathrm{a}_k^{(j)} = 0$. Furthermore, the number of new PTs $\mathrm{n}_k^{\mathrm{n}}$ can be obtained from $\overline{\mathbf{r}}_k$ as $\mathrm{n}_k^{\mathrm{n}} = \sum_{j=1}^{\mathrm{m}_k} \overline{\mathrm{r}}_k^{(j)}$. Using these relations and (19), it follows that there is a one-to-one relation between $(\mathbf{a}_k, \overline{\mathbf{r}}_k, \underline{\boldsymbol{\uptheta}}_k, \mathrm{n}_k^{\mathrm{n}}, \mathrm{n}_k^{\mathrm{c}})$ and $(\mathbf{a}_k, \overline{\mathbf{r}}_k, \mathrm{m}_k)$, and thus we have (cf. (5))

$$p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k, \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}|\underline{\boldsymbol{y}}_k) = p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k, m_k|\underline{\boldsymbol{y}}_k). \tag{20}$$

For now, we consider only legacy PT-measurement associations $\boldsymbol{a}_k$ and new PT-measurement associations $\overline{\boldsymbol{r}}_k$ that satisfy Assumption A4 from [1, Sec. I-C]. We recall that this assumption postulates that a measurement cannot originate from more than one target (modeled by a legacy PT or a new PT) simultaneously and a target can generate at most one measurement. We have (cf. (6))

$$p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k, \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}} | \underline{\boldsymbol{y}}_k) = p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k | \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k)\, p(\underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}} | \underline{\boldsymbol{y}}_k)$$
$$= p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k | \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k)\, p(\underline{\boldsymbol{\theta}}_k | \underline{\boldsymbol{y}}_k)\, p(n_k^{\mathrm{n}})\, p(n_k^{\mathrm{c}})\,. \tag{21}$$

Here, in the second step, we used Assumption A6 from [1, Sec. I-C] and Assumption Vu4 from [1, Sec. VIII-A], both in the corrected form provided in Section 1. That is, we assumed that $\underline{\boldsymbol{\theta}}_k$, $\mathsf{n}_k^{\mathrm{n}}$, and $\mathsf{n}_k^{\mathrm{c}}$ are conditionally independent given $\underline{\boldsymbol{y}}_k$, and thus $p(\underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}} | \underline{\boldsymbol{y}}_k) = p(\underline{\boldsymbol{\theta}}_k | \underline{\boldsymbol{y}}_k)\, p(n_k^{\mathrm{n}} | \underline{\boldsymbol{y}}_k)\, p(n_k^{\mathrm{c}} | \underline{\boldsymbol{y}}_k)$; and that $\mathsf{n}_k^{\mathrm{n}}$ and $\mathsf{n}_k^{\mathrm{c}}$ are independent of $\underline{\boldsymbol{y}}_k$, and thus $p(n_k^{\mathrm{n}} | \underline{\boldsymbol{y}}_k) = p(n_k^{\mathrm{n}})$ and $p(n_k^{\mathrm{c}} | \underline{\boldsymbol{y}}_k) = p(n_k^{\mathrm{c}})$. We recall that $p(\underline{\boldsymbol{\theta}}_k | \underline{\boldsymbol{y}}_k)$ is given by (16) and (17), $p(n_k^{\mathrm{n}})$ is given by (18), and $p(n_k^{\mathrm{c}})$ is given by (3).

It remains to find an expression of $p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k | \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k)$. We have

$$p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k | \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k) = p(\overline{\boldsymbol{r}}_k | \boldsymbol{a}_k, \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k)\, p(\boldsymbol{a}_k | \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k)\,. \tag{22}$$

Let us first determine the second factor, $p(\boldsymbol{a}_k | \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k)$. For given $\underline{\boldsymbol{\theta}}_k$, n_k^{n}, n_k^{c}, and $\underline{\boldsymbol{y}}_k$, the number $N'_{\boldsymbol{a}_k}$ of legacy PT-measurement associations $\boldsymbol{a}_k$ can be found similarly to (7), i.e.,

$$N'_{\boldsymbol{a}_k} = \frac{m_k!}{\big(m_k - n_k^{\mathrm{d}}(\underline{\boldsymbol{\theta}}_k)\big)!} = \frac{m_k!}{(n_k^{\mathrm{n}} + n_k^{\mathrm{c}})!}\,,$$

where (19) was used. Assuming that given $\underline{\boldsymbol{\theta}}_k$, n_k^{n}, n_k^{c}, and $\underline{\boldsymbol{y}}_k$, each legacy PT-measurement association $\boldsymbol{a}_k$ is equally likely, we thus obtain

$$p(\boldsymbol{a}_k | \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k) = \frac{1}{N'_{\boldsymbol{a}_k}} = \frac{(n_k^{\mathrm{n}} + n_k^{\mathrm{c}})!}{m_k!}\,. \tag{23}$$

Next, we determine the first factor in (22), $p(\overline{\boldsymbol{r}}_k | \boldsymbol{a}_k, \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k)$. For given $\boldsymbol{a}_k$, $\underline{\boldsymbol{\theta}}_k$, n_k^{n}, n_k^{c}, and $\underline{\boldsymbol{y}}_k$, the number $N_{\overline{\boldsymbol{r}}_k}$ of new PT-measurement associations $\overline{\boldsymbol{r}}_k$ is equivalent to the number of draws of n_k^{n} measurements out of $m_k - n_k^{\mathrm{d}}(\underline{\boldsymbol{\theta}}_k)$ measurements without replacement and without respecting the drawing order. (Note that the drawing order is of no relevance here since draws differing only in the drawing order result in the same vector $\overline{\boldsymbol{r}}_k$.) Hence, $N_{\overline{\boldsymbol{r}}_k}$ is given by

$$N_{\overline{\boldsymbol{r}}_k} = \binom{m_k - n_k^{\mathrm{d}}(\underline{\boldsymbol{\theta}}_k)}{n_k^{\mathrm{n}}} = \frac{\big(m_k - n_k^{\mathrm{d}}(\underline{\boldsymbol{\theta}}_k)\big)!}{n_k^{\mathrm{n}}!\,\big(m_k - n_k^{\mathrm{d}}(\underline{\boldsymbol{\theta}}_k) - n_k^{\mathrm{n}}\big)!} = \frac{(n_k^{\mathrm{n}} + n_k^{\mathrm{c}})!}{n_k^{\mathrm{n}}!\, n_k^{\mathrm{c}}!}\,.$$

Again assuming that each of these draws is equally likely, we obtain

$$p(\overline{\boldsymbol{r}}_k | \boldsymbol{a}_k, \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k) = \frac{1}{N_{\overline{\boldsymbol{r}}_k}} = \frac{n_k^{\mathrm{n}}!\, n_k^{\mathrm{c}}!}{(n_k^{\mathrm{n}} + n_k^{\mathrm{c}})!}\,. \tag{24}$$

Inserting (23) and (24) into (22) then yields

$$p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k | \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}}, \underline{\boldsymbol{y}}_k) = \frac{n_k^{\mathrm{n}}!\, n_k^{\mathrm{c}}!}{m_k!}\,. \tag{25}$$

Finally, inserting (25), (16), (18), and (3) into (21) yields

$$p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k, \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}} | \underline{\boldsymbol{y}}_k) = \frac{n_k^{\mathrm{n}}!\, n_k^{\mathrm{c}}!}{m_k!}\, \frac{\mu_{\mathrm{n}}^{n_k^{\mathrm{n}}}}{n_k^{\mathrm{n}}!}\, e^{-\mu_{\mathrm{n}}}\, \frac{\mu_{\mathrm{c}}^{n_k^{\mathrm{c}}}}{n_k^{\mathrm{c}}!}\, e^{-\mu_{\mathrm{c}}} \prod_{j=1}^{j_{k-1}} p\big(\underline{\theta}_k^{(j)} | \underline{\boldsymbol{y}}_k^{(j)}\big)$$

$$= \frac{e^{-\mu_{\mathrm{n}}} \mu_{\mathrm{n}}^{n_k^{\mathrm{n}}} e^{-\mu_{\mathrm{c}}} \mu_{\mathrm{c}}^{n_k^{\mathrm{c}}}}{m_k!} \prod_{j=1}^{j_{k-1}} p\big(\underline{\theta}_k^{(j)} \big| \underline{\boldsymbol{y}}_k^{(j)}\big) \,,$$

and inserting (17) further leads to

$$p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k, \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}} | \underline{\boldsymbol{y}}_k) = \frac{e^{-\mu_{\mathrm{n}}} \mu_{\mathrm{n}}^{n_k^{\mathrm{n}}} e^{-\mu_{\mathrm{c}}} \mu_{\mathrm{c}}^{n_k^{\mathrm{c}}}}{m_k!} \left(\prod_{j:\, \underline{\theta}_k^{(j)}=1} \underline{r}_k^{(j)} p_{\mathrm{d}}\big(\underline{\boldsymbol{x}}_k^{(j)}\big)\big) \right) \prod_{j':\, \underline{\theta}_k^{(i)}=0} \big(1 - \underline{r}_k^{(j')} p_{\mathrm{d}}\big(\underline{\boldsymbol{x}}_k^{(j')}\big)\big) \,. \tag{26}$$

Let us introduce the index set $\mathcal{D}_{\boldsymbol{a}_k} \triangleq \big\{ j \in \{1, \ldots, j_{k-1}\} : a_k^{(j)} \neq 0 \big\}$, which represents the set of detected PTs, and the index set $\mathcal{N}_{\overline{\boldsymbol{r}}_k} \triangleq \big\{ m \in \{1, \ldots, m_k\} : \overline{r}_k^{(m)} = 1 \big\}$, which represents the set of new PTs. Here, we recall that $a_k^{(j)} = 0$ indicates that legacy PT j is not detected and $a_k^{(j)} \neq 0$ indicates that it is detected. Hence, $j \notin \mathcal{D}_{\boldsymbol{a}_k}$ is equivalent to $\underline{\theta}_k^{(j)} = 0$ and $j \in \mathcal{D}_{\boldsymbol{a}_k}$ to $\underline{\theta}_k^{(j)} = 1$. Furthermore, $\overline{r}_k^{(m)} = 1$ indicates that new PT m is detected, and $\overline{r}_k^{(m)} = 0$ that it is not detected. We thus have $n_k^{\mathrm{d}}(\boldsymbol{\theta}_k) = |\mathcal{D}_{\boldsymbol{a}_k}|$ and $n_k^{\mathrm{n}} = |\mathcal{N}_{\overline{\boldsymbol{r}}_k}|$. Using these facts along with the identity (cf. (19)) $n_k^{\mathrm{c}} = m_k - n_k^{\mathrm{d}}(\boldsymbol{\theta}_k) - n_k^{\mathrm{n}}$, Eq. (26) can be rewritten as

$$p(\boldsymbol{a}_k, \overline{\boldsymbol{r}}_k, \underline{\boldsymbol{\theta}}_k, n_k^{\mathrm{n}}, n_k^{\mathrm{c}} | \underline{\boldsymbol{y}}_k) = \frac{e^{-\mu_{\mathrm{n}}} \mu_{\mathrm{n}}^{|\mathcal{N}_{\overline{\boldsymbol{r}}_k}|} e^{-\mu_{\mathrm{c}}} \mu_{\mathrm{c}}^{m_k - |\mathcal{D}_{\boldsymbol{a}_k}| - |\mathcal{N}_{\overline{\boldsymbol{r}}_k}|}}{m_k!} \left(\prod_{j \in \mathcal{D}_{\boldsymbol{a}_k}} \underline{r}_k^{(j)} p_{\mathrm{d}}\big(\underline{\boldsymbol{x}}_k^{(j)}\big) \right) \prod_{j' \notin \mathcal{D}_{\boldsymbol{a}_k}} \big(1 - \underline{r}_k^{(j')} p_{\mathrm{d}}\big(\underline{\boldsymbol{x}}_k^{(j')}\big)\big) \,. \tag{27}$$

So far, we considered only legacy PT-measurement associations $\boldsymbol{a}_k$ and new PT-measurement associations $\overline{\boldsymbol{r}}_k$ that satisfy Assumption A4 from [1, Sec. I-C]. To extend expression (27) to arbitrary $\boldsymbol{a}_k \in \{0, 1, \ldots, m_k\}^{j_{k-1}}$ and $\overline{\boldsymbol{r}}_k \in \{0, 1\}^{m_k}$, we multiply it by the indicator functions $\psi(\boldsymbol{a}_k)$ (see [1, Secs. IV-B and VIII-D]) and $\prod_{m \in \mathcal{N}_{\overline{\boldsymbol{r}}_k}} \Gamma_{\boldsymbol{a}_k}^{(m)}$ (see [1, Sec. VIII-D]), which enforce Assumption A4. Finally, using (20), we obtain [1, Eq. (56)].

References

[1] F. Meyer, T. Kropfreiter, J. L. Williams, R. A. Lau, F. Hlawatsch, P. Braca, and M. Z. Win, "Message passing algorithms for scalable multitarget tracking," *Proc. IEEE*, vol. 106, no. 2, pp. 221–259, Feb. 2018.

[2] Y. Bar-Shalom, P. K. Willett, and X. Tian, *Tracking and Data Fusion: A Handbook of Algorithms*. Storrs, CT: Yaakov Bar-Shalom, 2011.